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Boundary value problems for a mixed type equation of the 3rd order with variable coefficients

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Abstract. The existence and uniqueness of the solution to the boundary value problem for a third-order mixed-type equation with variable coefficients in the lower terms is proven, with conditions for the gluing of the function itself and its first- and second-order derivatives on the line $y = 0$, where the type of equation changes when a second-order mixed parabolic-hyperbolic operator is applied to a first-order linear differential operator with constant coefficients. By reducing the order of the equation, the problem was reduced to the Tricomi problem for a second-order mixed parabolic-hyperbolic equation with continuous conditions for the function itself and its first-order derivative with respect to y on the line of change of the equation type. By the method of elimination of the system of equations obtained from the parabolic and hyperbolic parts of the domains, the solvability of the problem was reduced to the solvability of the Fredholm integral equation of the second kind. A sufficient condition for the solvability of the integral equation was obtained through estimates of the kernel of the equation. The solution of the problem was split into two problems in the regions under consideration: in the parabolic part of the region, the first boundary value problem for the heat conduction equation was solved using the Green's function method, and in the hyperbolic part of the region, bounded by the characteristics of the equation and the line $y = 0$, the solution of the problem using the Riemann function construction method was determined as the solution of the Cauchy problem. By applying a curvilinear integral, the solution to the problem in the areas under consideration was found. The necessity of the requirement of continuity of the function itself and its first two derivatives with respect to y on the line of change of the equation type was justified. Sufficient conditions for the unique classical solvability of the boundary value problem were established. The obtained conditions for the solvability of the boundary value problem provided a theoretical basis for the development of numerical methods for solving applied problems in aerohydrodynamics, geophysics, and engineering thermodynamics

Keywords: existence; uniqueness; Green's function; Riemann's function; integral equation; order reduction method

Introduction

Mathematical modeling of a number of applied problems in fluid mechanics, physics, and mathematical biology has led to the formulation and investigation of the solvability of boundary value problems with nonlocal and integral conditions for third-order mixed-type equations, since in practice, the measurement of certain characteristics of the desired function and its derivatives is possible only in averaged or integral form.

The formulation of well-posed boundary value problems for partial differential equations and their

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investigation are urgent tasks of the modern theory of partial differential equations. Despite the wide application of mixed-type equations in problems of gas dynamics, hydrodynamics, and other applied disciplines, boundary value problems for third-order mixed-type equations with two independent variables remain insufficiently studied, which necessitates further research in this direction.

K. Belakroum [1] studied a nonlocal boundary value problem for third-order partial differential equations in Hilbert space with a self-adjoint positively defined operator, in which stability estimates for the solution of two nonlocal problems for third-order partial differential equations were obtained.

In the works of R.R. Ashurov & Yu.E. Fayziev [2], the uniqueness and existence of the solution of the inverse problem of determining the order of the fractional derivative with respect to time in an inhomogeneous subdiffusion equation with an arbitrary elliptic differential operator with constant coefficients in an n-dimensional torus was proved. Using the classical Fourier method, it was proved that the value of the solution at a fixed moment in time, based on observational data, uniquely reconstructs the order of the fractional derivative.

Zh.A. Balkizov [3] investigated a boundary value problem with shift for a third-order inhomogeneous parabolic-hyperbolic equation with a wave operator in the hyperbolic region, when a linear combination of the values of the desired function on two independent characteristics and on the line of type change is given as a boundary condition. Necessary and sufficient conditions for the existence and uniqueness of a regular solution to the problem were found.

N.K. Ochilova & T.K. Yuldashev [4] investigated the existence and uniqueness of the solution of a nonlocal boundary value problem for a degenerate differential equation of mixed type. A parabolic-hyperbolic equation with a Gerasimov-Caputo fractional derivative was considered. The uniqueness of the solution was proved by the method of energy integrals using some properties of hypergeometric functions and integro-differential operators of fractional order. The existence of a solution is proved by the method of integral equations.

The solvability in anisotropic Sobolev spaces of nonlocal boundary value problems for third-order pseudoparabolic equations was studied by A.I. Kozhanov & G.I. Tarasova [5]. A feature of the problems studied is that they impose a condition on the spatial variable that combines the generalised Samarskii-Ionkin condition and an integral type condition. The purpose of the work was to prove the existence and uniqueness of regular solutions to the problems studied – solutions having all generalised Sobolev derivatives included in the corresponding equation.

In the work of D.K. Durdieva & Sh.B. Merajova [6], an inverse problem related to finding an unknown right-hand side was studied for a mixed parabolic-hyperbolic equation with a Bessel operator. Based on the method of separation of variables, the problem is reduced to solving ordinary differential equations with respect to the coefficients of the expansion into Bessel-Fourier series of unknown functions in terms of orthonormal Bessel functions of the first kind of zero order. A criterion for the uniqueness and existence of a solution to the posed problem was established.

In the work of S.N. Sidorov [7], an initial-boundary value problem was investigated for an inhomogeneous mixed parabolic-hyperbolic equation of three variables with a degenerating parabolic part in a rectangular parallelepiped. A criterion for the uniqueness of the solution was established. The solution was constructed as a sum of an orthogonal series. The stability of the solution with respect to the boundary function and the right-hand side of the equation was established.

In the work of R.Kh. Makaova [8], a theorem on the existence and uniqueness of a regular solution to a mixed boundary value problem for a third-order hyperbolic equation with order degeneration inside the domain was proved. In the positive part of the domain, the considered equation coincided with the Aller equation, which is a third-order hyperbolic equation. And in the negative part of the domain, it coincided with a degenerate hyperbolic equation of the first kind. The uniqueness of the solution of the studied problem was proved by the Tricomi method.

In the work of D.K. Durdiev [9], direct and inverse problems for a model mixed parabolic-hyperbolic equation were considered. In the direct problem, an analog of the Tricomi problem for this equation with a characteristic line of type change was considered. The unknown in the inverse problem is the variable coefficient at the lower term of the parabolic equation. For its determination with respect to the solution defined in the parabolic part of the domain, an integral overdetermination condition is given. Local theorems on the unique solvability of the posed problems in the sense of a classical solution are proved.

The purpose of this work was to formulate and investigate new well-posed boundary value problems for third-order equations, when a mixed parabolic-hyperbolic operator is applied to a first-order differential operator of a certain form. The questions of where and what mandatory boundary conditions should be set for a well-posed problem, the number of gluing conditions on the line of change of the equation type, and what methods should be used to solve the problem were unknown beforehand, so they were determined during the study of the problem.

Materials and Methods

Let C^{n+m} means the class of functions having all continuous derivatives $\partial^{r+s}/\partial x^r \partial y^s (r = 0,1,..., n;\ s = 0,1,..., m)$. In this work, a boundary value problem is considered where a mixed parabolic-hyperbolic operator is applied to a linear first-order differential operator.

Problem Statement. In the domain D, limited by line segments AC: $x+y=0$, CB: $x-y=l$, BB_0: $x=l$, B_0A_0: $y=h$, B_0A: $x=0(l,h>0)$, the equation is considered:

$$L_1L_2u=0, \quad (1)$$

$$L_1\equiv\begin{cases}\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial y}+c_1(x,y), y>0,\\ \frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}+a_2(x,y)\frac{\partial}{\partial x}+b_2(x,y)\frac{\partial}{\partial y}+c_2(x,y), y<0,\end{cases}$$

$$L_2\equiv\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y},$$

where $a_2(x,y)$, $b_2(x,y)$, $c_1(x,y)$, $c_2(x,y)$ – given functions satisfying the following smoothness conditions:

$$c_1(x,y)\in C(\overline{D}_1), \forall(x,y)\in\overline{D}_1: c_1(x,y)\le 0,$$

$$a_2(x,y), a_{2x}(x,y), a_{2y}(x,y),$$
$$b_2(x,y), b_{2x}(x,y), b_{2y}(x,y), c_2(x,y)\in C(\overline{D}_2). \quad (2)$$

Let $D_1=D\cap(y>0)$, $D_2=D\cap(y<0)$. The characteristic equation for equation (1) in the domain D_1 has the form $-(dy)^3+(dy)^2dx=0$. Consequently, the line $y=const$ is a 2-fold characteristic, $x-y=const$ a single characteristic of equation (1) in the domain D_1. The characteristic equation for equation (1) in the domain D_2 has the form $-(dy)^3+(dy)^2dx+dy(dx)^2-(dx)^3=0$, which can be written as $(dx+dy)(dx-dy)^2=0$. Consequently, the line $x-y=const$ is a 2-fold characteristic, and $x+y=const$ a single characteristic of equation (1) in the domain D_2. Thus, equation (1) in the domain D_1 belongs to the hyperbolic type, and in the domain D_2 – also hyperbolic type. This fact means that equation (1) is a mixed-type equation in the domain D, since when crossing the lines $y=0$ the characteristics of equation (1) differ, which affects the well-posedness of the problem. This paper considers the formulation and investigation of the well-posedness of the following problem.

Problem 1. It is required to determine the function $u(x,y)$ with the following properties:

1) $u(x,y)\in C(\overline{D})\cap C^2(D)\cap[C^{3+2}(D_1)\cup C^{3+3}(D_2)]$;
2) $u(x,y)$ is a solution of equation (1) in the domain $D\setminus(y=0)$;
3) $u(x,y)$ satisfies the following boundary conditions:

$$u(0,y)=\varphi_1(y), u(l,y)=\varphi_2(y), 0\le y\le h, \quad (3)$$

$$u_x(0,y)=\varphi_3(y), u_x(l,y)=\varphi_4(y), 0\le y\le h, \quad (4)$$

$$\left.\frac{\partial u}{\partial n}\right|_{AC}=\psi(x), 0\le x\le\frac{l}{2}, \quad (5)$$

where $\varphi_i(y)(i=\overline{1,4})$, $\psi(x)$ – given smooth functions, n – inner normal, and:

$$\varphi_i(y)\in C^2[0,h](i=1,2), \varphi_j(y)\in C^1[0,h]$$
$$(j=3,4), \psi(x)\in C^2\left[0,\frac{l}{2}\right], \quad (6)$$

$$\varphi_3(0)-\varphi_1'(0)=-\sqrt{2}\psi(0). \quad (7)$$

From the statement of Problem 1, the following gluing conditions follow:

$$u(x,-0)=u(x,+0)=\tau(x), u_y(x,-0)=$$
$$=u_y(x,+0)=\nu(x), 0\le x\le l,$$
$$u_{yy}(x,-0)=u_{yy}(x,+0)=\mu(x), 0\le x\le l, \quad (8)$$

where $\tau(x)$, $\nu(x)$, $\mu(x)$ – currently unknown functions.

By the method of reducing the order of equations, Problem 1 is reduced to an analogue of the Tricomi problem for a mixed parabolic-hyperbolic equation with continuous gluing conditions with the line of change of type $y=0$ in the following way. The solution to Problem 1 was considered separately in each of the domains. D_1 and D_2. Equation (1) in the domain D_1 is written as a system of equations:

$$\begin{cases}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\upsilon_1(x,y), (x,y)\in D_1,\\ \frac{\partial^2\upsilon_1}{\partial x^2}-\frac{\partial\upsilon_1}{\partial y}+c_1(x,y)\upsilon_1=0, (x,y)\in D_1,\end{cases} \quad (9)$$

and in the domain D_2 – in the form of:

$$\begin{cases}\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=\upsilon_2(x,y), (x,y)\in D_2,\\ \frac{\partial^2\upsilon_2}{\partial x^2}-\frac{\partial^2\upsilon_2}{\partial y^2}+a_2(x,y)\frac{\partial\upsilon_2}{\partial x}+b_2(x,y)\frac{\partial\upsilon_2}{\partial y}+\\ \quad +c_2(x,y)\upsilon_2=0, (x,y)\in D_2.\end{cases} \quad (10)$$

From the gluing condition (8) on the line $y=0$ the equalities follow: $\upsilon_1(x,+0)=\tau'(x)-\nu(x)$, $\upsilon_2(x,-0)=\tau'(x)-\nu(x)$. Therefore, according to the first two conditions (8), the equality holds: $\upsilon_1(x,-0)=\upsilon_2(x,+0)$. By differentiating the first equation of system (9) with respect to y the equality is obtained: $\frac{\partial^2u}{\partial x\partial y}-\frac{\partial^2u}{\partial y^2}=\frac{\partial\upsilon_1(x,y)}{\partial y}$. Then at $y=0$ taking into account the gluing condition (8) the equality follows: $\frac{\partial^2u(x,+0)}{\partial x\partial y}-\frac{\partial^2u(x,+0)}{\partial y^2}=\frac{\partial\upsilon_1(x,+0)}{\partial y}$, which can be written as: $\frac{\partial\upsilon_1(x,+0)}{\partial y}=\nu'(x)-\mu(x)$. Similarly, from the first equation of system (10), the equality is obtained: $\frac{\partial\upsilon_2(x,-0)}{\partial y}=\nu'(x)-\mu(x)$. Consequently, according to the second and third conditions of (8), the equality holds: $\frac{\partial\upsilon_1(x,+0)}{\partial y}=\frac{\partial\upsilon_2(x,-0)}{\partial y}$. Thus, for functions $\upsilon_1(x,y)$ and $\upsilon_2(x,y)$ the following gluing conditions hold on the line $y=0$:

$$\upsilon_1(x,+0)=\upsilon_2(x,-0)=\nu_1(x), \frac{\partial\upsilon_1(x,+0)}{\partial y}=\frac{\partial\upsilon_2(x,-0)}{\partial y}=\mu_1(x), \quad (11)$$

where $\nu_1(x)$ and $\mu_1(x)$ – currently unknown functions. The functions $\nu_1(x)$ and $\mu_1(x)$ are related to the functions $\tau(x)$, $\nu(x)$ and $\mu(x)$ as follows:

$$\nu_1(x)=\tau'(x)-\nu(x), \mu_1(x)=\nu'(x)-\mu(x). \quad (12)$$

Thus, the equations

$$\frac{\partial^2\upsilon_1}{\partial x^2}-\frac{\partial\upsilon_1}{\partial y}+c_1(x,y)\upsilon_1=0, (x,y)\in D_1, \quad (13)$$

$$\frac{\partial^2 \upsilon_2}{\partial x^2} - \frac{\partial^2 \upsilon_2}{\partial y^2} + a_2(x,y)\frac{\partial \upsilon_2}{\partial x} + b_2(x,y)\frac{\partial \upsilon_2}{\partial y} + \\ + c_2(x,y)\upsilon_2 = 0, (x,y) \in D_2, \quad (14)$$

are related by the conjugation conditions (11). Therefore, to determine the functions $\upsilon_1(x,y)$ and $\upsilon_2(x,y)$ the following auxiliary problem was considered.

Problem 2. It is required to determine the function $\upsilon_1(x,y) \in C(\overline{D}_1) \cap C^1(\overline{D}_1) \cap C^{2+1}(D_1)$ and $\upsilon_2(x,y) \in C(\overline{D}_2) \cap$ $\cap C^1(\overline{D}_2) \cap C^2(D_2)$, satisfying the following conditions:

1) $\upsilon_1(x,y)$ is a solution of equation (13) in the domain. D_1, $\upsilon_2(x,y)$ is a solution of equation (14) in the domain D_2;

2) $\upsilon_1(x,y)$ satisfies the boundary conditions:

$$\upsilon_1(0,y) = \tilde{\varphi}_1(y),\ \upsilon_2(l,y) = \tilde{\varphi}_2(y),\ 0 \le y \le l, \quad (15)$$

3) $\upsilon_2(x,y)$ satisfies the boundary condition

$$\upsilon_2(x,-x) = \tilde{\psi}(x),\ 0 \le x \le \frac{l}{2}, \quad (16)$$

4) $\upsilon_1(x,y)$ *and* $u\upsilon_2(x,y)$ satisfy the gluing conditions (11), where $\tilde{\varphi}_1(y) = \varphi_3(y) - \varphi_1'(y)$, $\tilde{\varphi}_2(y) = \varphi_4(y) - \varphi_2'(y)$, $\tilde{\psi}(x) = -\sqrt{2}\psi(x)$.

To solve Problem 2, it is first necessary to find the functions $\nu_1(x)$ and $\mu_1(x)$. To determine these functions, relationships obtained from both domains are required D_1, as well as from the domain D_2. To obtain a relationship from the domain D_1 between $\nu_1(x)$ and $\mu_1(x)$ a limiting transition from the equation is used (13) at $y \to +0$, and to obtain the second relationship between $\nu_1(x)$ and $\mu_1(x)$. The general representation of the solution to the Cauchy problem for equation (14), presented through Riemann functions, is used [10].

Results and Discussion

First, Problem 2 was considered. To solve Problem 2, it is necessary to obtain relationships derived from both regions D_1, and also from the domain D_2.

Obtaining a relationship from the domain D_1. When tending towards y to $+0$ from equation (13), a relationship between the functions follows. $\nu_1(x)$ and $\mu_1(x)$, from equation (13), a relationship between the functions follows. D_1:

$$\nu_1''(x) + c_1(x,0)\nu_1(x) - \mu_1(x) = 0, 0 \le x \le l. \quad (17)$$

From the boundary conditions (15), the following conditions are obtained:

$$\nu_1(0) = \tilde{\varphi}_1(0),\ \nu_1(l) = \tilde{\varphi}_2(0). \quad (18)$$

If introduce the notation:

$$\nu_1(x) = \tilde{\varphi}_1(0) + \frac{x}{l}[\tilde{\varphi}_2(0) - \tilde{\varphi}_1(0)] + \nu_2(x), \quad (19)$$

where $\nu_2(x)$ – a new unknown function, then from (17) the relation is obtained:

$$\nu_2''(x) + c_1(x,0)\,\nu_2(x) = \mu_1(x) + \Phi_1(x),\ 0 \le x \le l, \quad (20)$$

where $\Phi_1(x) = -c_1(x,\ 0)\{\tilde{\varphi}_1(0) + \frac{x}{l}\ [\tilde{\varphi}_2(0) - \tilde{\varphi}_1(0)]\}$. The boundary conditions are then of the form:

$$\nu_2(0) = 0,\ \nu_2(l) = 0. \quad (21)$$

The following lemma holds.

Lemma 1. If the conditions are met: $\forall x \in [0,\ l]$: $c_1(x,\ 0) \le 0$, (22) then the homogeneous problem (20)-(21) has only a trivial solution.

Proof. After multiplying the homogeneous equation $\nu_2''(x) + c_1(x,\ 0)\ \nu_2(x) = 0$ at $\nu_2(x)$ and of integrating the obtained relation from 0 to l taking into account the homogeneous conditions (21) the equality holds:

$$\int_0^l \nu_2(x)[\nu_2''(x) + c_1(x,\ 0)\ \nu_2(x)]dx = \\ = \int_0^l \{-[\nu_2'(x)]^2 + c_1(x,\ 0)[\nu_2(x)]^2\}dx = 0.$$

It is obvious that if the condition is met (22), then $\forall x \in [0, l]$: $\nu_2(x) \equiv 0$. Lemma 1 is proved. Solution of the inhomogeneous equation (20), satisfying the homogeneous boundary conditions (21), has the form:

$$\nu_2(x) = \Phi_2(x) + \int_0^l G(x,\ \xi)\ \mu_1(\xi)d\xi,\ 0 \le x \le l, \quad (22)$$

where $\Phi_2(x) = \int_0^l G(x,\ \xi)\ \Phi_1(x)dx$, $G(x,\ \xi)$ – Green's function. Then equality (19) will be written as:

$$\nu_1(x) = \Phi(x) + \int_0^l G(x,\ \xi)\ \mu_1(\xi)d\xi,\ 0 \le x \le l, \quad (23)$$

where $\Phi(x) = \tilde{\varphi}_1(0) + \frac{x}{l}\ [\tilde{\varphi}_2(0) - \tilde{\varphi}_1(0)] + \Phi_2(x)$. Equality (23) represents the relationship between functions $\nu_1(x)$ and $\mu_1(x)$ obtained from the domain D_1.

Obtaining a relationship from the domain D_2. For the hyperbolic equation (14) in the domain D_2. The Cauchy problem with conditions is considered:

$$\upsilon_2(x,\ 0) = \nu_1(x),\ \upsilon_{2y}(x,\ 0) = \mu_1(x),\ 0 \le x \le l, \quad (24)$$

the solution of which is presented in the form:

$$\upsilon_2(x,y) = \frac{1}{2}[R(x,y;\ x+y,\ 0)\ \nu_1(x+y) + R(x,y;\ x-y,\ 0) \times \\ \times \nu_1(x-y)] + \frac{1}{2}\int_{x+y}^{x-y} R_\eta(x,y;\ \xi,\ 0) + b_2(\xi,\ 0)\ R(x,y;\ \xi,\ 0)]\nu_1(\xi) \times \\ \times d\xi - \frac{1}{2}\int_{x+y}^{x-y} R(x,y;\ \xi,\ 0)\ \mu_1(\xi)d\xi, \quad (25)$$

where $R(x,\ y;\ \xi,\ \eta)$ – Riemann function [11]. This function is defined as the solution to the Goursat problem for the equation:

$$R_{\xi\xi} - R_{\eta\eta} - (a_2R)_\xi - (b_2R)_\eta + c_2R = 0, \quad (26)$$

satisfying the conditions:

$$R(x,y;\xi,x+y-\xi) = \\ = exp\left\{-\frac{1}{2}\int_\xi^x [a_2(t,x+y-t) + b_2(t,x+y-t)]\,dt\right\}, (27)$$

$$R(x,y;\xi,\xi-x+y)=$$
$$=exp\left\{\frac{1}{2}\int_x^{\xi}[a_2(t,t-x+y)-b_2(t,t-x+y)]\,dt\right\}, \quad (28)$$

$$R(x,y;x,y)=1. \quad (29)$$

After using condition (16), equality (28) is written as:

$$R(x,-x;2x,0)\nu_1(2x)=2\tilde{\psi}(x)-R(x,-x;0,0)\nu_1(0)-$$
$$-\int_0^{2x}[R_\eta(x,-x;\xi,0)+b_2(\xi,0)R(x,-x;\xi,0)]\,\nu_1(\xi)d\xi+$$
$$+\int_0^{2x}R(x,-x;\xi,0)\,\mu_1(\xi)d\xi,\ 0\le x\le\frac{l}{2}. \quad (30)$$

Obviously, that $\forall x\in[0,\frac{l}{2}]$: $0\le 2x\le l$. Taking into account that $\nu_1(0)=\tilde{\psi}(0)$, then from (30), upon substitution $2x=z$ and then z at x the relation is obtained:

$$R\left(\frac{x}{2},-\frac{x}{2};x,0\right)\nu_1(x)=$$
$$=-\int_0^x\left[R_\eta\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)+b_2(\xi,0)R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)\right]\times$$
$$\times\nu_1(\xi)d\xi+\int_0^x R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)\mu_1(\xi)d\xi+$$
$$+2\tilde{\psi}\left(\frac{x}{2}\right)-R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)\tilde{\psi}(0). \quad (31)$$

Lemma 2. $\forall x\in[0,l]$ the inequality holds:

$$R\left(\frac{x}{2},-\frac{x}{2};x,0\right)>0. \quad (32)$$

Proof. Let $\xi=x-y$. Then equality (28) will take the form $R(x,y;x-y,0)=exp\left\{\frac{1}{2}\int_x^{x-y}\begin{bmatrix}a_2(t,t-x+y)-\\-b_2(t,t-x+y)\end{bmatrix}dt\right\}$. If $y=-x$, then this equality will be written as $R(x,-x;2x,0)=exp\left\{\frac{1}{2}\int_x^{2x}[a_2(t,t-2x)--b_2(t,t-2x)]\,dt\right\}$. When replacing $2x$ to z, then z to x, the following relation holds $R\left(\frac{x}{2},-\frac{x}{2};x,0\right)=exp\left\{\frac{1}{2}\int_{\frac{x}{2}}^{x}[a_2(t,t-x)-b_2(t,t-x)]\,dt\right\}$. From this equality, the validity of inequality (32) follows. Lemma 2 is proved.

If take into account inequality (32) from Lemma 2, then it follows from (31) that the relation between functions $\nu_2(x)$ and $\mu_2(x)$, obtained from the domain D_2, is presented as:

$$\nu_1(x)=\int_0^x P_1(x,\xi)\,\nu_1(\xi)d\xi+$$
$$+\int_0^x P_2(x,\xi)\,\mu_1(\xi)d\xi+\psi_1(x), \quad (33)$$

where $P_1(x,\xi)=-\frac{\left[R_\eta\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)+b_2(\xi,0)R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)\right]}{R\left(\frac{x}{2},-\frac{x}{2};x,0\right)}$.

$$\frac{P_2(x,y)=R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)}{R\left(\frac{x}{2},-\frac{x}{2};x,0\right)},\quad \psi_1(x)=\frac{\left[2\tilde{\psi}\left(\frac{x}{2}\right)-R\left(\frac{x}{2},-\frac{x}{2};0,0\right)\tilde{\psi}(0)\right]}{R\left(\frac{x}{2},-\frac{x}{2};x,0\right)}.$$

After inverting the Volterra part of equation (33), the relation is obtained:

$$\nu_1(x)=\int_0^x K_1(x,\xi)\,\mu_1(\xi)d\xi+\Psi_1(x), \quad (34)$$

where $K_1(x,\xi)=P_2(x,\xi)+\int_\xi^x G(x,t)\,P_2(t,\xi)dt$, $\Psi_1(x)=\psi_1(x)+\int_0^x G(x,t)\,\psi_1(t)dt$, $G(x,\xi)$ – resolvent of the kernel $P_1(x,\xi)$. Equality (34) represents the relationship between the functions $\nu_1(x)$ and $\mu_1(x)$, obtained from the domain D_2.

Reduction of the problem to an integral equation. After eliminating $\nu_1(x)$ From equations (23) and (34), the following relationship is obtained:

$$\int_0^x K_1(x,\xi)\,\mu_1(\xi)d\xi=\int_0^l G(x,\xi)\mu_1(\xi)d\xi+\Psi_2(x), \quad (35)$$

where $\Psi_2(x)=\Phi(x)-\Psi_1(x)$. After differentiating equation (35), the following equation is obtained:

$$K_1(x,x)\mu_1(x)+\int_0^x K_{1x}(x,\xi)\,\mu_1(\xi)d\xi=$$
$$=\int_0^l G_x(x,\xi)\mu_1(\xi)d\xi+\Psi_2'(x). \quad (36)$$

This is $\forall x\in[0,l]$: $K_1(x,x)=P_2(x,x)=1$, then equation (36) is written as:

$$\mu_1(x)=\int_0^x[-K_{1x}(x,\xi)]\mu_1(\xi)d\xi+$$
$$+\int_0^l G_x(x,\xi)\mu_1(\xi)d\xi+\Psi_2'(x). \quad (37)$$

After inverting the Volterra part of equation (37), a Fredholm integral equation of the second kind is obtained:

$$\mu_1(x)=\Psi(x)+\int_0^l K(x,\xi)\,\mu_1(\xi)d\xi, \quad (38)$$

where $K(x,\xi)=G_x(x,\xi)+\int_0^x R_2(x,t)\,G_x(t,\xi)dt$, $\Psi(x)=\Psi_2'(x)+\int_0^x R_2(x,t)\,\Psi_2'(t)dt$, and $R_2(x,t)$ – resolvent of the kernel $-K_{1x}(x,\xi)$. Let $\|K\|_{C(\bar{Q})}=\max_{(x,\xi)\in\bar{Q}}|K(x,\xi)|$, where $Q=\{(x,\xi):0\le x\le l,0\le\xi\le l\}$. If:

$$l\cdot\|K\|_{C(\bar{Q})}<1, \quad (39)$$

then the integral equation (38) has a unique solution [12]. The solution of equation (38) can be represented as:

$$\mu_1(x)=\Psi(x)+\int_0^l R(x,\xi)\,\Psi(\xi)d\xi, \quad (40)$$

where $R(x,\xi)$ – resolvent of the kernel $K(x,\xi)$. Then from (23) is also determined $\nu_1(x)$:

$$\nu_1(x)=\Phi(x)+\int_0^l G(x,\xi)\Psi(\xi)d\xi+$$
$$+\int_0^l G(x,\xi_1)d\xi_1\int_0^l R(\xi_1,\xi)\Psi(\xi)d\xi. \quad (41)$$

Then the function $\upsilon_2(x,y)$, presented by formula (25), is fully determined, since the functions $\nu_1(x)$ and $\mu_1(x)$ are determined by formulas (40) and (41), respectively.

Solution of the problem in the domain D_1. In the domain D_1. The following problem is considered.

Problem 3. Find in the domain D_1 the solution of equation (13), satisfying the conditions:

$$\upsilon_1(0,y)=\tilde{\varphi}_1(y),\ \upsilon_1(l,y)=\tilde{\varphi}_2(y),$$
$$0\le y\le l;\ \upsilon_1(x,0)=\nu_1(x),0\le x\le l,$$

where $\nu_1(x)$ determined by the formula (41), and $\nu_1(0)=\tilde{\varphi}_1(0)$, $\nu_1(l)=\tilde{\varphi}_2(0)$

By the Green's function method, the solution of Problem 3 is reduced to solving a Volterra integral equation of the second kind:

$$\upsilon_1(x,y) = \upsilon_0(x,y) + \int_0^y d\eta \int_0^l N(x,y;\xi,\eta)\, \upsilon_1(\xi,\eta)d\xi, \quad (42)$$

where $(x, y; \xi, \eta) = -c_1(\xi, \eta)\ G(x, y; \xi, \eta)$, $G(x, y; \xi, \eta)= =\frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{+\infty}\left\{exp\left[-\frac{(x-\xi+2nl)^2}{4(y-\eta)}\right] - exp\left[-\frac{(x+\xi+2nl)^2}{4(y-\eta)}\right]\right.$ – Green's function [13], and $\upsilon_0(x, y) = \int_0^y G\xi(x, y; 0, \eta)\ \tilde{\varphi}_1(\eta)d\eta - \int_0^y G\xi(x, y; l, \eta)\ \tilde{\varphi}_2(\eta)d\eta + \int_0^l G(x, y; \xi, 0)\ \nu_1(\xi)d\xi$ – known function. Since the kernel $N(x,y;\xi,\eta)$ has a weak singularity, therefore equation (42) has a unique solution, consequently Problem 3 is uniquely solvable in the domain D_1.

Problem 4. Find the solution to the equation in the domain

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \upsilon_1(x,y),\ (x,y) \in D_1, \quad (43)$$

satisfying conditions (3).

For solving Problem 4, a curvilinear integral of the second kind was used. On the plane $\xi O\eta$ the domain D_1 is divided into two regions: $D_1 = D_{11} \cup D_{12}$, where $D_{11}=\{(\xi,\eta): 0<\xi<l, \xi<\eta<l\}$, $D_{12}=\{(\xi,\eta): 0<\xi<l, 0<\eta<\xi\}$. In the domain D_{11} an arbitrary point is chosen $M_1(x,y)$ and a straight line is drawn $\eta=\xi-x+y$ through this point, which is parallel to the bisector $\eta=\xi$ of the first coordinate angle. This line intersects the y-axis at point $N_1(0, y-x)$. Further, on the segment N_1M_1 a curvilinear integral of the second kind is applied to equation (43):

$$\int_0^x \left[u_\xi(\xi,\xi-x+y) + u_\eta(\xi,\xi-x+y)\right] d\xi = \\ = \int_0^x \upsilon_1(\xi,\xi-x+y)d\xi.$$

This is $u_\xi(\xi, \xi-x+y) + u_\eta(\xi, \xi-x+y) = \frac{\partial}{\partial\xi}\, u(\xi, \xi-x+y)$, then after integrating the left side of the equality, taking into account the first boundary condition (3), the following relation holds:

$$u(x,y) = \varphi_1(y-x) + \int_0^x \upsilon_1(\xi, \xi-x+y)d\xi,\ (x,y) \in D_{11}.$$

Similarly, in the domain $D_{12}=\{(\xi,\eta): 0<\xi<l, 0<\eta<\xi\}$ an arbitrary point is considered $M_2(x, y)$ and a line $\eta=\xi-x+y$, parallel to the bisector $\eta=\xi$, which intersects the line $\xi=l$ at the point $N_2(l, l-x+y)$. Next, a curvilinear integral of the 2nd kind from the equation is calculated (43) on the segment M_2N_2:

$$\int_x^l \left[u_\xi(\xi,\xi-x+y) + u_\eta(\xi,\xi-x+y)\right] d\xi = \\ = \int_x^l \upsilon_1(\xi,\xi-x+y)d\xi.$$

Hence, as in the previous case, the solution to Problem 1, satisfying the second boundary condition (3), is determined as follows:

$$u(x,y) = \varphi_2(l-x+y) - \int_x^l \upsilon_1(\xi, \xi-x+y)d\xi,\ (x,y) \in D_{12}.$$

Thus, the solution to Problem 2 is presented as:

$$u(x,y) = \\ = \begin{cases} \varphi_1(y-x) + \int_0^x \upsilon_1(\xi,\xi-x+y)\,d\xi, (x,y) \in D_{11}, \\ \varphi_2(l-x+y) - \int_x^l \upsilon_1(\xi,\xi-x+y)\,d\xi, (x,y) \in D_{12}. \end{cases} \quad (44)$$

It should be noted that the solution to problem 4 in the field of D_1 may have a discontinuity of the first kind on the line $y=x$. If compliance with the agreement condition is required

$$\varphi_2(l) = \varphi_1(0) + \int_0^l \upsilon_1(\xi, \xi)d\xi, \quad (45)$$

then $u(x,y) \in C(\bar{D}_1)$ from (44). The trace of the function is defined on the segment AB:

$$\tau(x) = \varphi_2(l-x) - \int_x^l \upsilon_1(\xi, \xi-x)d\xi,\ 0 \le x \le l. \quad (46)$$

Solving problems in the field of D_2. In the domain D_2 considering the following problem.

Problem 5. Find in the domain D_2 equation solution

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \upsilon_2(x,y),\ (x,y) \in D_2, \quad (47)$$

satisfying the condition $u(x, 0) = \tau(x)$, $0 \le x \le l$.

Let $M_3(x, y)$. An arbitrary point of the region D_2. Through the point $M_3(x, y)$ the direct line is being built $\eta=\xi-x+y$, which is parallel to the line CB: $x-y=l$. This line intersects the x-axis at the point $N_3(x-y, 0)$. On the line $M_3N_3=\{(x,y): x<\xi<x-y, \eta=\xi-x+y\}$. The line integral of the second kind is applied to equation (47):

$$\int_x^{x-y}[u_\xi(\xi, \xi-x+y) + u_\eta(\xi, \xi-x+y)]d\xi = \int_x^{x-y}\upsilon_1(\xi, \xi-x+y)d\xi. \quad (48)$$

Considering that $u_\xi(\xi, \xi-x+y) + u_\eta(\xi, \xi-x+y) = =\frac{\partial}{\partial\xi}u(\xi, \xi-x+y)$, from equality (48), the solution to problem 5 is determined, representable in the form:

$$u(x,y) = \tau(x-y) - \int_x^{x-y}\upsilon_2(\xi, \xi-x+y)d\xi,\ (x,y) \in D_2. \quad (49)$$

Hence, the following theorem holds.

Theorem 1. Let the conditions be met (5), (6), (29), (32) and (45). Then the solution to Problem 1 exists and is unique. If condition (45) is not met, then the solution to Problem 1 in the domain D_1 suffers discontinuities of the first kind on the line $y=x$. Therefore, condition (45) provides a sufficient condition for the continuity of the desired solution along the line $y=x$.

Example 1. Let $\forall y \in [0, h]$: $\tilde{\varphi}_1(y) = \varphi_3(y) - \varphi_1'(y) \equiv 0$, $\tilde{\varphi}_2(y) = \varphi_4(y) - \varphi_2'(y) \equiv 0$, $\tilde{\psi}(x) =-\sqrt{2}\psi(x) \equiv 0$. Therefore, $\forall x \in [0, l]$: $\varphi_3(y) = \varphi_1'(y)$, $\varphi_4(y) = \varphi_2'(y)$, $\forall x \in [0, \frac{l}{2}]$: $\psi(x) = 0$. Then problem 2 has a trivial solution, that is $\forall (x,y) \in \bar{D}_1$: $\upsilon_1(x,y) \equiv 0$, $\forall (x,y) \in \bar{D}_2$: $\upsilon_2(x,y) \equiv 0$. From formula (44), the solution of Problem 1 is determined in the form:

$$u(x,y) = \begin{cases} \varphi_1(y-x), 0 \le x < y, 0 \le y \le h, \\ \varphi_2(l-x+y), y < x \le l, 0 \le y \le h. \end{cases} \quad (50)$$

It is obvious that if $\varphi_1(0) = \varphi_2(l)$, then $u(x,y) \in C(\bar{D}_1)$. It follows from (50) that $\tau(x) = \varphi_2(l-x)$. Solution of Problem 1 in the domain D_2 is represented by the formula (49):

$$u(x,y)=\tau(x-y)=\varphi_2(l-x+y),\ (x,y)\in D_2.$$

Thus, the existence and uniqueness of the solution to Problem 1 for a third-order equation have been proven, where a mixed parabolic-hyperbolic second-order operator is applied to a first-order linear differential operator. Various formulations of boundary value problems for a third-order equation with two independent variables were considered in the work of B.I. Islomov & O.Kh. Abdullaev [13], where the unique solvability of a nonlocal problem with an integral gluing condition for a third-order equation with a parabolic-hyperbolic operator was proven.

This operator includes the Caputo fractional derivative and a nonlinear term involving the trace of the solution $u(x, 0)$. Unlike this work, the equation under consideration is a third-order equation in which a first-order differential operator with coefficients a, b, and c acts on a second-order parabolic-hyperbolic operator.

A.N. Mironov [14] considered the third-order equation with a dominant partial derivative (Bianchi equations), providing the formulation of the Darboux problem and the definition of the Riemann–Hadamard function. Relying on the possibility of representing the Riemann function explicitly for one class of third-order Bianchi equations equivalent by function, sufficient conditions on the coefficients of the Bianchi equation were proposed to ensure the construction of the Riemann–Hadamard function in terms of hypergeometric functions.

In the work by Zh.A. Balkizov *et al*. [15], a boundary value problem with displacement was studied for a nonhomogeneous third-order parabolic-hyperbolic type equation, where one of the boundary conditions is given as a linear combination of the values of the sought function on independent characteristics. The work established necessary and sufficient conditions for the existence and uniqueness of a regular solution to the problem. It was shown that if the necessary conditions on the given functions found in the work are violated, the corresponding homogeneous problem has infinitely many linearly independent solutions, and the set of solutions to the corresponding nonhomogeneous problem may exist only under additional requirements on the given functions.

In the work of R.Kh. Makaova [16], a mixed boundary value problem was studied for a nonhomogeneous third-order hyperbolic-type Aller equation. Using the method of separation of variables, the theorem on existence and uniqueness of a regular solution was proven. A representation of the Green's function was obtained. The explicit form of the regular solution to the problem under study was written out.

A.N. Mironov & A.P. Volkov [17] proved the existence and uniqueness of the solution to a boundary value problem with conditions on one of the characteristics and on a free line for a system of hyperbolic equations with multiple characteristics. An analogue of the Riemann–Hadamard method was developed for the stated problem, and the definition of the Riemann–Hadamard matrix was given. The solution to the problem was constructed in terms of the introduced Riemann–Hadamard matrix.

A.K. Urinov & K.S. Khalilov [18] formulated and studied a nonclassical problem with an integral condition for a third-order parabolic-hyperbolic equation. The unique solvability of the posed problem was proven using the method of integral equations. Moreover, the posed problem was equivalently reduced to a problem for a second-order parabolic-hyperbolic equation with an unknown right-hand side. In the study of this latter problem, formulas for solving the Cauchy problem for a hyperbolic equation with a singular coefficient and a spectral parameter, as well as solutions of the first boundary value problem for the parabolic Fourier equation, were used.

In the article by O.Kh. Abdullaev & T.K. Yuldashev [19], the existence and uniqueness of solutions to inverse problems with a nonlinear gluing condition for a loaded parabolic-hyperbolic type equation were investigated. The problem was reduced to the study of a nonlinear Fredholm integral equation of the second kind. The theorem on existence and uniqueness of the solution was proved by the method of successive approximations.

O.M. Dzhokhadze *et al*. [20] studied a mixed problem with Dirichlet and Poincaré boundary conditions for a second-order hyperbolic equation and systems. In the linear case, an explicit representation of the solution was given, and questions of uniqueness and solvability of the posed problem were also investigated depending on the nature of the nonlinearities present in the system.

The work of A. Matchanova [21] is devoted to the solution of a local problem for a third-order parabolic-hyperbolic equation with a Caputo fractional derivative. The considered problem includes the third boundary condition in the parabolic domain and a continuity condition on the line $y = 0$. The existence of the solution was proven using the theory of Volterra-type integral equations.

A.I. Kozhanov & G.R. Ashurova [22] investigated the well-posedness of inverse problems of determining, together with the solution, a degenerate differential equation with multiple characteristics and an unknown coefficient defining the external influence (free term). The nature of degeneracy in the studied equation, as well as the form of the unknown coefficient, are determined by the time variable. For the studied problems, theorems on the existence and uniqueness of regular solutions having all generalised derivatives according to S.L. Sobolev entering the equation are proven.

M.G. Beshtokov [23] considered initial-boundary value problems for a fractional-order moisture transfer equation with a nonlocal linear source and variable coefficients. Assuming the existence of a regular solution for each of the considered first and third initial-boundary value problems, an a priori estimate in differential form was obtained. Estimates in difference form and convergence of the solution of each difference problem were obtained. Numerical calculations illustrating the obtained theoretical results were performed.

Yu.P. Apakov & A.A. Sopuev [24] proved the existence of a unique solution for nonlocal conjugation problems in a rectangular domain for a third-order partial differential equation, when for $y > 0$ the characteristic equation has three multiple roots, and for $y < 0$ it has one simple and two multiple roots. Using the Green function and the method of integral equations, the solution of the problems is equivalently reduced to the solution of a boundary value problem for the trace of the sought function at $y = 0$, and then to the solution of a Fredholm integral equation of the second kind, the solvability of which is proved by the method of successive approximations. The solution of the problem for $y > 0$ is constructed by the Green function method, and for $y < 0$ – by reducing the problem to a two-dimensional Volterra integral equation of the second kind.

It should be noted that the cited works considered various formulations of boundary value problems for a third-order equation with two independent variables. In the present work, one of the variants of the boundary value problem formulation for a third-order equation was studied, where a mixed parabolic-hyperbolic second-order operator with a line of type change is involved $y = 0$, is applied to a first-order linear differential operator.

Conclusions

The unique solvability of the main problem 1 for equation (1) is proved when the mixed parabolic-hyperbolic operator is applied on the left to a first-order differential operator. In solving problem 1, the method of reducing the order of the equation was used, as a result of which this problem is reduced to problem 2 (an analogue of the Tricomi problem) for a second-order mixed parabolic-hyperbolic equation with a line of type change $y = 0$. The peculiarity of this problem is that this line is a characteristic of equation (1) in the domain D_1. Using the methods of integral equations, Green's function, and Riemann function, the existence and uniqueness of the solution to problem 2, when the equation in the domain D_1 belongs to the parabolic type, and in the domain D_2 – of hyperbolic type. A key point in solving Problem 2 was finding the trace of the sought function and its derivative on the line $y = 0$. Next, the solution of problem 2 was decomposed into solutions of the first boundary value problem for equation (13) in the domain D_1, and in the domain D_2 – to the solution of the Cauchy problem for the hyperbolic type equation (14). Then, problems 4 and 5 were considered. The peculiarity of the solution to problem 4 was that on the line $y = x$ the domain D_1. The sought solution undergoes a first kind discontinuity. Therefore, it is additionally required to satisfy the continuity condition of the sought solution along the line $y = x$, as a result, the continuity of the function $u(x, y)$, as the solution of problem 4 in the domain D_1 is provided. When the coefficients at the lower-order terms of the equation are absent, the continuity of the solution is ensured by the fulfillment of the compatibility condition: $\varphi_1(0) = \varphi_2(l)$. Thus, the set goal in solving the main problem 1 has been achieved. The indicated research method is also applicable if the gluing conditions have first-kind discontinuities on the line of type change.

It should also be noted that there is a need to study, in the direction of this research topic, the case when the line of change of the type of the equation is a line $x = 0$ since in this case, unlike in the present study, this line is not a characteristic of either the parabolic or the hyperbolic equation.

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Conflict of Interest

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Аннотация. Төмөнкү мүчөлөрү өзгөрмөлүү коэффициенттер болгон үчүнчү тартиптеги аралаш типтеги теңдеме үчүн функциянын өзүнүн жана анын биринчи жана экинчи тартиптеги туундулары үчүн теңдеменин түрү өзгөрө турган $y = 0$ сызыгында жабыштыруу шарттары орун алган учурдагы чек аралык маселенин чечиминин бар экендиги жана жалгыздыгы далилденген. Бул маселе экинчи тартиптеги аралаш параболалык-гиперболалык оператор биринчи тартиптеги турактуу коэффициенттүү сызыктуу дифференциалдык операторго колдонулган учурда каралат. Теңдеменин тартибин төмөндөтүү жолу менен каралып жаткан маселе функциянын өзүн жана анын биринчи тартиптеги туундусун теңдеменин түрү өзгөрө турган $y = 0$ сызыгында жабыштыруу шарттары менен экинчи тартиптеги аралаш параболалык-гиперболалык теңдеме үчүн Трикоми маселесине келтирилет. Областтардын параболалык жана гиперболалык бөлүктөрүнөн алынган теңдемелер системасын жоюу ыкмасы менен маселенин чечилиши экинчи түрдөгү Фредгольмдун интегралдык теңдемесинин чечилүүчүлүгүнө келтирилет. Интегралдык теңдеменин чечилиши үчүн жетиштүү шарт теңдеменин ядросун баалоо аркылуу алынат. Маселени чечүү каралып жаткан аймактарда эки маселеге бөлүнөт: аймактын параболалык бөлүгүндө жылуулук теңдемеси үчүн биринчи чек аралык маселе Грин функциясы ыкмасы менен чыгарылат, ал эми теңдеменин мүнөздөөчү сызыктары жана $y = 0$ сызыгы менен чектелген аймакта гиперболалык теңдеме үчүн Римандын функциясын куруу ыкмасы менен маселени чечүү Коши маселесин чечүүгө алып келинет. Ийри сызыктуу интегралды колдонуу менен каралып жаткан аймактардагы маселенин чечими табылган. Теңдеменин түрүнүн $y = 0$ өзгөрүү сызыгында функциянын өзүнүн жана анын биринчи жана экинчи тартиптеги туундуларынын үзгүлтүксүздүгүн талап кылуу зарылчылыгы негизделген. Чек аралык маселесинин классикалык чечиминин жашашы үчүн жетиштүү шарттар табылган. Краевая маселенин чечилиш шарттары аэрогидродинамика, геофизика жана инженердик жылуулук техникасы тармагындагы колдонмолук маселелерди сандык ыкмалар менен чечүүнүн теориялык негизин түзөт

Негизги сөздөр: чечимдин жашашы; чечимдин жалгыздыгы; Гриндин функциясы; Римандын функциясы; интегралдык тендеме; тартибин төмөндөтүү методу

Краевые задачи для уравнения смешанного типа 3-го порядка с переменными коэффициентами

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Аннотация. Доказывается существование и единственность решения краевой задачи для уравнения смешанного типа третьего порядка с переменными коэффициентами при младших членах с условиями склеивания самой функции и её производных первого и второго порядков на линии $y = 0$ изменения типа уравнения, когда смешанный параболо-гиперболический оператор второго порядка применяется к линейному дифференциальному оператору первого порядка с постоянными коэффициентами. Методом понижения порядка уравнения задача сводилась к задаче Трикоми для уравнения смешанного параболо-гиперболического типа второго порядка с непрерывными условиями склеивания самой функции и её производной первого порядка по y на линии изменения типа уравнения. Методом исключения системы уравнений, полученных из параболической и гиперболической части областей, разрешимость задачи сводилась к разрешимости интегрального уравнения Фредгольма второго рода. Получено достаточное условие разрешимости интегрального уравнения через оценки ядра уравнения. Решение задачи расщеплялось на две задачи в рассматриваемых областях: в параболической части области методом функции Грина решалась первая краевая задача для уравнения теплопроводности, а в гиперболической части области, ограниченная характеристиками уравнения и линией $y = 0$, решение задачи методом построения функции Римана определялась как решение задачи Коши. Применением криволинейного интеграла найдено решение задачи в рассматриваемых областях. Обосновывалась необходимость требования непрерывности самой функции и её первых двух производных по y на линии изменения типа уравнения. Установлены достаточные условия для однозначной классической разрешимости краевой задачи. Полученные условия разрешимости краевой задачи обеспечили теоретическую основу для разработки численных методов решения прикладных задач в аэрогидродинамике, геофизике и инженерной теплотехнике

Ключевые слова: существование; единственность; функция Грина; функция Римана; интегральное уравнение; метод понижения порядка