e-ISSN 1694–8645

Journal of Osh State University. Mathematics. Physics. Technical Sciences

Publisher:
Osh State University

Year of foundation: 2022

Frequency: semi-annual

State registration:
Registration certificate of the Ministry of Justice of the Kyrgyz Republic
No. 10296 dated 15.06.2022.

The journal is presented in the following international scientometric databases, repositories and scientific systems:
eLIBRARY.RU; Litmaps; Crossref

Editorial office address:
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
E-mail: info@mpht-oshsu.com
https://mpht-oshsu.com/en

e-ISSN 1694–8645

Ош мамлекеттик университетинин Жарчысы. Математика. Физика. Техника

Негиздөөчү:
Ош мамлекеттик университети

Негизделген жылы: 2022

Чыгаруунун мезгилдүүлүгү: жылына 2 жолу

Мамлекеттик каттоо:
Кыргыз Республикасынын Юстиция министрлигинин каттоо күбөлүгү
№10296 15.06.2022.

Журнал төмөнкү эл аралык илимий-метрикалык маалымат базаларында, репозиторийлерде жана издөө системаларында камтылган:
eLIBRARY.RU; Litmaps; Crossref

Редакциянын дареги:
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
E-mail: info@mpht-oshsu.com
https://mpht-oshsu.com/kg

e-ISSN 1694–8645

Вестник Ошского государственного университета. Математика. Физика. Техника

Издатель:
Ошский государственный университет

Год основания: 2022

Периодичность выпуска: 2 раза в год

Государственная регистрация:
Регистрационное свидетельство Министерства юстиции Кыргызской Республики
№ 10296 от 15.06.2022.

Журнал представлен в таких международных наукометрических базах данных, репозитариях и поисковых системах:
eLIBRARY.RU; Litmaps; Crossref

Адрес редакции:
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
E-mail: info@mpht-oshsu.com
https://mpht-oshsu.com/ru

Редакциялык коллегия

Contents / Мазмуну / Содержание

VOL. 4, No. 1
Journal of Osh State University.
Mathematics. Physics. Technical Sciences
2025
Journal of Osh State University
Journal homepage: https://mpht-oshsu.com/en

UDC 511.3
DOI: 10.52754/16948645_2025_4(1)_8

Article's History:
Received 20.02.2025 Revised 08.05.2025 Accepted 19.06.2025

Proving the correctness of the Collatz hypothesis

Anatoliy Cherkasenko*
Deputy Company Commander for Moral and Psychological Support
60th Mechanized Brigade of the Armed Forces of Ukraine
https://orcid.org/0009-0000-7947-8549

Abstract. Proof of the correctness of the Collatz conjecture is topical research, as it represents one of the many unsolved problems in mathematics. Understanding the properties of this sequence has important implications for other areas of mathematics, such as number theory or graph theory. The aim of the study was to prove the Collatz hypothesis as a theorem. The research methodology included the analysis of numerical sequences, the use of mathematical induction, recursive, combinatorial methods and computer modelling. The study analysed the properties of sequences generated by the Collatz hypothesis, particularly their recursive properties. The study determined that each odd number has a unique "potential" that affects the behaviour of the sequence. The correlation between even and odd numbers in the context of the hypothesis, as well as the influence of division and multiplication operations on the change of number sequences, are investigated. The results of the study showed that sequences according to the Collatz hypothesis have specific patterns that can be used to develop effective approaches to their proof. The study also determined that the operations of dividing by 2 multiplying by 3 and adding 1 have a systemic effect on the development of the sequence. The results of the study showed that the proposed method of studying sequences helped to determine the correct location of numbers in an infinite sequence of natural numbers *n* and other groups of numbers. The main difference of the proposed approach is the introduction of the concept of "potential of an odd number" and "blocks of numbers" related to this odd number. The potential of an odd number was a property of numbers that confirmed the hypothesis and was used to call the Collatz problem a theorem. The practical significance of the study lies in the possibility of applying new methods of analysing numerical sequences in computer science, cryptography and other fields requiring optimisation of computing processes

Keywords: natural numbers; odd number potential; mathematical induction; sequence analysis; open problem

Introduction

The hypothesis proposed by Lothar Collatz is one of the key unsolved problems in mathematics, which has attracted interest and remained open for many decades. It considers a sequence of operations on a positive integer n. According to the hypothesis, choose any initial positive integer n and apply the following rules: if n is even, divide it by 2 ($n / 2$); if n is odd, multiply it by 3 and add 1 ($3n + 1$), a new number is obtained. The process is repeated for the resulting number, and the hypothesis states that, regardless of the initial choice of n, the number 1 is always reached in the end. The solution to this problem can be a significant contribution to the development of mathematical science. Proving Collatz's conjecture as a theorem requires a systematic and detailed approach to analysing number sequences and studying their properties. There is a need to carefully consider all variants and develop a mathematical logic for its convincing proof, which creates a difficult challenge for researchers.

O.V. Zelensky *et al.* [1] analysed various counterexamples of the Collatz hypothesis. The author investigated the aspect of the minimal counterexample and presented proofs of several examples. P. Kosobutskyy & V. Karkulovskyy [2] conducted a study on the repetition and structuring of $3n + 1$ transformation sequences as arguments in support of the Collatz conjecture. The authors demonstrated that the absence of infinity of odd numbers in a subsequence is not an argument

Suggested Citation:
Cherkasenko A. Proving the correctness of the Collatz hypothesis. J Osh State Univ Math Phys Tech Sci. 2025;4(1):8–18. DOI: 10.52754/16948645_2025_4(1)_8

*Corresponding author

against the Collatz conjecture. Instead, this property is a universal characteristic of the sequence of transformations for natural numbers using the $3n+1$ algorithm. The study also determined that there is a recurrent relationship between the parameters for the sequences of the Collatz transforms of any pair of positive integers n and $2n$.

According to S.V. Kmita [3], the Collatz hypothesis naming has a reason – no one has been able to prove it so far. Collatz put forward this hypothesis in 1937 (according to other sources, in 1928 or 1932), and since then, many attempts have been made to verify or refute this statement using purely mathematical proofs. However, all those mathematicians have been able to achieve is an experimental test of the hypothesis.

P. Kosobutskyy & D. Rebot [4] analysed the Collatz conjecture, considering it a binomial problem similar to Newton's problem. They demonstrated that in the opposite direction, the Collatz sequence consists of the lower bounds of the corresponding cycles, and the last element tends to become a multiple of three for odd numbers. The researchers also determined that for infinite cycles isolated from the main graph with minimum amplitudes of 5, 7, and 17, additional conditions arise that regulate their lower limits of oscillation.

According to O. Leshchenko [5], one of the key problems that arise when studying the Collatz hypothesis is the issue of randomness in the behaviour of this sequence, which seems to have a randomly given nature. In the paper, the author revealed the importance of this problem for mathematics and considered the possibility of its application in the modern world. Some key characteristics of the Collatz sequence were analysed using the Maple computer algebra system. A hypothesis was put forward that as the initial number increases, the maximum length of the Collatz sequence grows no faster than the logarithmic function, which was confirmed by numerical calculations. E. Dyachenko [6] presented a proof of the Collatz conjecture, also known as the $3x+1$ problem. The proof was based on the number systems of rational bases, their modifications and the sequence of ordered numerical intervals. The latter provided a new way of thinking about integers. The proof was obtained by dividing the numbers into ordered intervals.

Notably, some aspects of Collatz's conjecture remain unexplored, in particular, the possibility of complex cyclic structures when changing the condition, for example, $3n+1$ to $3n-1$, or $5n+1$ or $5n-1$, which are not reduced to 1, and the consideration of sequence properties for large numbers. Addressing these questions may help to better understand the nature of the Collatz hypothesis and its possible applications. The study aimed to develop a new approach to the study and analysis of the Collatz sequence. Using the concept of odd number potential and blocks of numbers allows for a deeper and more systematic analysis of this sequence.

Materials and Methods

The study focused on analysing the behaviour of numerical sequences formed using the rules of the Collatz hypothesis. The main research methods used were mathematical analysis, combinatorics, algebraic methods, mathematical induction and formalisation of results using mathematical logic. The first stage of the study was a detailed analysis of the number sequences generated by the $3n+1$ rule. The properties of odd numbers and their relationships with even numbers were analysed. Their dualistic properties were investigated, and their relationships were established in the context of the Collatz hypothesis. The mathematical properties of sequences of numbers and their influence on the behaviour of numbers during iterations were analysed. Particular attention was devoted to the study of odd numbers, since they, after the $3n+1$ operation, form more complex structures than even numbers, which rapidly decrease due to division by 2. The behaviour of numbers during iterations was analysed and some regularities were revealed the rapid decrease of numbers and the possibility of long sequences of odd numbers before reaching 1. The possibility of studying the structure of complex sequences to understand the general laws of the Collatz hypothesis was also considered.

The method of mathematical induction was used to analyse the recursive properties of the Collatz function. This analysis determined the diversity of sequence behaviour, including fast and slow convergence to 1. The base case was established for the number 1, which is the endpoint for all possible sequences. Then, an inductive assumption was built to prove that if the hypothesis is true for the number n, it is also true for the number $n+1$. This formalised the process of proving the hypothesis for a wide class of natural numbers.

Combinatorial methods were used to analyse the number of steps required to reach 1 depending on the initial number n. This was used not only to estimate the length of the sequences but also to establish certain patterns in the frequency of occurrence of certain numbers in the sequences. This was used to study the cyclic properties of sequences and to predict possible ways of developing numbers depending on the initial conditions, substantially improving understanding of their structure and convergence.

Algebraic methods were used to analyse the structure of Collatz sequences and their relationship with other algebraic objects, such as groups or rings. This was used to consider the Collatz sequences as part of the general algebraic structure and to determine how algebraic properties can affect the behaviour of numbers. To further determine the arithmetic properties of numbers in Collatz sequences, various results from number theory, such as the prime number theorem or the prime factorisation theorem, were used. This was used to explore the characteristics of numbers in sequences and their relationship to the structure of primes in more depth. The last stage of the study was the formalisation of the results using mathematical logic. For this purpose, predicative logic and set theory were used to carefully structure the proofs and convincingly prove the theorem within the framework of formal mathematical systems.

Results

Collatz's conjecture, also known as the $3n + 1$ problem or Hayes' problem, is one of the most famous unsolved problems in mathematics. The Collatz conjecture states that no matter what initial positive integer is chosen, sooner or later the process will end, and the number 1 will be reached. Despite its simplicity, this problem has attracted many scientists because of its unpredictability and lack of proof, as the process itself ends in an infinite loop: 1, 4, 2, 1, 4... The Collatz hypothesis itself did not attract the attention of the scientific community for a long time, until in the 1950s Helmud Hasse (Syracuse University) formulated it as a mathematical game, thus relating the problem to numerical sequences. When American amateur mathematician Martin Gardner formulated it as a mathematical puzzle in 1972, interest in the Collatz conjecture grew significantly, and later Stanislav Ulam, Hungarian mathematician Pal Erdős, and others tried to solve it. With the involvement of powerful computing resources, the Collatz conjecture was confirmed for numbers of order 10 to the 28th power. However, no algorithm was found to prove the correctness of the hypothesis, so the problem is open for proof.

Despite many attempts and tests for different initial values of n to disprove the hypothesis, no counterexamples have been found so far. That is, all known tests confirm that the Collatz hypothesis is valid [7-9]. Although the Collatz Conjecture is very simple to formulate and understand, its difficulty lies in the fact that there has been no mathematical proof of its validity for any positive integer n. This situation makes it one of the most famous open problems in mathematics. Many mathematicians have worked on this problem, and although some additional properties of this sequence have been found, the problem itself remains unsolved [10]. Mathematically, the Collatz conjecture was written as follows (1):

$$f(n) = \begin{cases} \frac{n}{2}, if\ n\ is\ even \\ 3n + 1, if\ n\ is\ odd \end{cases}. \tag{1}$$

Task 1. The analysis of the conditions of Hypothesis 1 shows the need to identify groups of numbers that participate in iterations when fulfilling the conditions of the task:

1. Even numbers. It is assumed that all even numbers that exist in nature and are used in iterations will lead to odd numbers.

2. Odd numbers. It is assumed that odd numbers have some specific properties they serve as the beginning of a cycle using the $3n + 1$ function.

3. Numbers $3n + 1$. These numbers are always even (odd number x odd number = odd number).

4. The numbers are $3n$; $3n = N_{even} - 1$.

This task aims to analyse the conditions of a hypothesis related to the structure of numbers during their iteration depending on their evenness or oddness.

The duality of the properties of odd numbers is noteworthy: each odd number $N_{j\text{-}odd}$ is the beginning of the cycle $3n + 1 = N_{even} - N_{even}/2^n = N_{j\text{-}odd}$. Analysis of the duality of the properties of odd numbers, it was concluded that each odd number serves as the beginning of a certain cycle that will lead to new odd numbers using the $3n+1$ function. This indicates that the numbers under study form a certain structure and sequence that can be investigated to obtain additional conclusions about their behaviour and properties.

The Collatz problem was considered an interesting game, and there is no need for practice in proving the hypothesis for scientific, technical and other spheres of life, at least not now. However, many examples of purely mathematical theories, which have nothing to do with practice, becoming an important tool for research and calculations, mathematical apparatus of theories in other scientific disciplines, show that proving the correctness (incorrectness) of a hypothesis is necessary, and the author offered a version of proving a hypothesis and turning it into a theorem.

The author suggested that there is a certain property that characterises the groups of numbers involved in the iterations, and this property determines whether the hypothesis is true. That is, it was assumed that if the groups of numbers involved in the iterations are correctly arranged, a hidden property of the numerical sequence of natural numbers N that determines the correctness/incorrectness of the Collatz hypothesis will be evident. The correct arrangement is the arrangement that shows the entire iteration process until its completion, i.e., until 1 and the infinite cycle 1, 4, 2, 1, 4... Below is the sequence of the author's proof of the Collatz conjecture as a theorem.

1. Lothar Collatz proposed the task: take any positive number N from the infinite sequence (2):

$$1,2,3,\ \dots\ \infty. \tag{2}$$

If even is returned – divide by 2. If it is odd, multiply by 3 and add 1 to the result. Do the same with the resulting number until 1 is the result. After that, an endless cycle is evident: 1, 4, 2, 1, 4... This is the Collatz hypothesis. No matter what the initial number is, the sequence comes to 1 and an infinite cycle.

2. To prove a hypothesis, a certain property in the presented infinite sequence of numbers that determines the correctness or incorrectness of the hypothesis should be determined.

3. The concept of "colour of number" (numbers) was introduced. The chosen property of a number, either due to its nature or as a result of predefined operations, as indicated by the assigned colour.

4. Even and odd number groups were considered. These two groups comprise all the numbers considered in the Collatz hypothesis (0 was not considered, but later it was discovered that it was impossible to do without 0). Even numbers are denoted by A and are black, odd numbers are denoted by B and are blue. In addition, the lilac C numbers, green X numbers, and red J numbers (defined below) were considered. There is also a group of numbers $3 \times B$ and a group of numbers $(3 \times B + 1)$.

5. The properties of these groups (sets) were analysed, they are infinite sets.

6. The odd numbers B have two properties according to the rules of the hypothesis:

a) each number B is the result of dividing an even number from the set of even numbers (3) by 2 the appropriate number of times:

$$Aand = Eand \times 2n, \tag{3}$$

where $n = 1,2,3, \dots \infty$; $i = 1,2,3, \dots \infty$;

b) each number B is the starting point for calculating an even number using formula (4):

$$3 \times E + 1. \tag{4}$$

7. Even numbers have the following properties according to the rules of the hypothesis:

a. each number A is a member of one of the sets defined by formula (3);

b. some of the numbers A are the result of the calculation using formula (4).

8. An infinite sequence of numbers was presented in the form of an infinite table (Table 1).

Table 1. An infinite sequence of even and odd numbers in the context of the Collatz hypothesis

K_i;n	0	1	2	3	4	5	6	...	n
0	1	2	4	8	16	32	64	...	1×2^n
1	3	6	12	24	48	96	192	...	3×2^n
2	5	10	20	40	80	160	320	...	5×2^n
3	7	14	28	56	112	224	448	...	7×2^n
4	9	18	36	72	144	288	576	...	9×2^n
5	11	22	44	88	176	352	704	...	11×2^n
6	13	26	52	104	208	416	832	...	13×2^n
7	15	30	60	120	240	480	960	...	15×2^n
...	...							...	
K_i	$1+2 \times K_i$	$(1+2 \times K_i) \times 2^n$						...	$(1+2 \times K_i) \times 2^n$

Note: K is the row index that defines the set of odd numbers in each row of the table
Source: compiled by the author

The second left column, starting from the second row, contains all odd blue numbers, the column is filled in using the formula (5):

$$Eand = 1 + 2 \times K_i, \tag{5}$$

where $i = 1,2,3, \dots \infty$; $C_i = 1,2,3, \dots \infty$. The leftmost column starts from the second row:

$$i = 1,2,3, \dots \infty,$$
$$C_i = 1,2,3, \dots \infty.$$

The top row starting from the second column is n:

$$n = 1,2,3, \dots \infty.$$

The lines after the blue numbers are filled with black (even) numbers according to the formula (6):

$$A_n = (1 + 2 \times K_i) \times 2n, \tag{6}$$

where $K_i = 0, 1, 2, 3 \dots \infty$; $n = 0, 1, 2, 3 \dots \infty$.

All combinations of K_i and n have been obtained (this means that Table 1 contains all natural numbers that exist in nature), i.e. the infinite sequence (2) and Table 1 contain the same numbers.

Next, number groups were analysed: $3 \times B$ and $3 \times B + 1$, $3 \times B$ always odd by definition. Each odd number in B is the starting point for calculating an even number using the formula $3 \times E + 1$. It was affirmed: $3 \times E_i = A - 1$, where $i = 1,2,3, \dots \infty$, it is problematic to denote A.

9. The number X is green (7):

$$X_i = A_i - 1, \tag{7}$$

where $i = 2, 4, 6 \dots \infty$.

Later, it was discovered that green numbers that are multiples of 3 and the result of division $X / 3$ are of interest. Number C has a lilac colour, lilac numbers are always even and odd (8):

$$C_i = X_i / 3, \tag{8}$$

where $i = 1, 3, 5, 7 \dots \infty$, if the result of dividing $X_i/3$ is not an integer, it does not turn lilac, such numbers were not considered.

Blue, black, green and lilac numbers were placed in Table 2. Green numbers not divisible by 3 were not highlighted.

Table 2. Classification of blue, green and lilac numbers in the context of the Collatz hypothesis

21	41	83	167	335	671	...	$(21\times 2^n)-1$
	42	84	168	336	672	...	21×2^n
19		25		101		...	$((19\times 2^n)-1)/3$
	37	75	151	303	607	...	$(19\times 2^n)-1$
	38	76	152	304	608	...	19×2^n
17	11		45		181	...	$((17\times 2^n)-1)/3$
	33	67	135	271	543	...	$(17\times 2^n)-1$
	34	68	136	272	544	...	17×2^n
15	29	59	119	239	479	...	$(1\times 2^n)-1$
	30	60	120	240	480	...	15×2^n
13		17		5		...	$((13\times 2^n)-1)/3$
	25	51	103	207	415	...	$(13\times 2^n)-1$
	26	52	104	208	416	...	13×2^n
11	7		29		117	...	$((11\times 2^n)-1)/3$
	21	43	87	175	351	...	$(11\times 2^n)-1$
	22	44	88	176	352	...	11×2^n
9	17	35	71	143	287	...	$(9\times 2^n)-1$
	18	36	72	144	288	...	9×2^n
7		9		37		...	$((7\times 2^n)-1)/3$
	13	27	55	111	223	...	$(7\times 2^n)-1$
	14	28	56	112	224	...	7×2^n
5	3		11		53	...	$((5\times 2^n)-1)/3$
	9	19	39	79	159	...	$(5\times 2^n)-1$
	10	20	40	80	160	...	5×2^n
3	5	11	23	47	95	...	$(3\times 2^n)-1$
	6	12	24	48	96	...	3×2^n
1		1		5		...	$((1\times 2^n)-1)/3$
	1	3	7	15	31	...	$(1\times 2^n)-1$
	2	4	8	16	32	...	1×2^n

Source: compiled by the author

10. Analysis of Table 2. The table shows all the blue (odd) numbers that exist in nature (9):

$$Eand = 1 + 2 \times K_i, \tag{9}$$

where $i = 0, 1, 2, 3\ldots\infty$.

Since it is impossible to avoid 0, formula (5) was used, and all odd numbers in nature can certainly be placed in Table 1. Table 2 shows all the green numbers (10) that exist in nature:

$$X_i = A_i - 1, \tag{10}$$

where $A_i = 2, 4, 6\ldots.\infty$.

For ease of reference, green numbers not divisible by 3 have not been removed from the table, and green numbers are odd. Table 2 shows all naturally occurring lilac numbers (11):

$$Y_i = X_i / 3, \tag{11}$$

where $i = 1, 3, 5\ldots.\infty$, lilac numbers are odd.

As can be seen from the previous step, for every blue number there is a corresponding lilac number of equal magnitude, and vice versa – for every lilac number there is a corresponding blue number of equal magnitudes. Notably, the blue and lilac odd numbers reflect the duality of the properties of odd numbers considered in the hypothesis. Each number B is the result of dividing an even number from the set of even numbers (12) by 2 the appropriate number of times:

$$An_i = B_i x2^n, \tag{12}$$

where $n = 0, 1, 2, 3\ldots.\infty$; $i = 0, 1, 2, 3\ldots.\infty$.

Each number B is the starting point for the calculation of an even number using the formula (13):

$$3xB + 1. \tag{13}$$

11. The concept of a block of odd blue numbers was introduced (it is emphasised that it is blue because all lilac and green numbers are also odd). An odd blue number block is an odd blue number and its corresponding rows: black even, green, lilac and red numbers (block 1, block 3, block 5, etc.).

12. The question was investigated: whether lilac and blue numbers can be placed in the same block.

The blue number (14):

$$Eand = (1 + 2 \times Kand),\ i = 0,1,2,3\ldots\infty. \tag{14}$$

Lilac number (15):

$$C_j = (1 + 2 \times K_j),\ j = 0,1,2,3 \dots \infty. \tag{15}$$

The numbers are equal (16, 17):

$$Eand = (1 + 2 \times Kand) = C_j = (1 + 2 \times K_j), \tag{16}$$

$$C_j = (1 + 2 \times K_j) = (((1 + 2 \times K_j) \times 2^n) - 1)/3, \tag{17}$$

where $n = 0, 1, 2, 3 \dots \infty$.

There is a unique solution to equation (12): $C_i = K_j = 0$; $n = 2$. Only one block 1 contains a blue 1 and a lilac 1, in all other blocks, it is impossible to place equal blue and lilac numbers in the same block.

13. To reflect the relationship between the lilac and blue numbers, the concept of block potential was introduced. It will be denoted by the letter J, the colour red. The potential of a blue number block is a property of a block determined by the duality of the odd numbers considered in the hypothesis, namely blue and lilac.

14. The block's potential is determined by the following rules:

a) the potential of the block is numbered with the numbers (18):

$$J = 1,2,3,4 \dots \infty; \tag{18}$$

b) the potential of block $J = 1$ is that of block 1. This is the lowest potential, the block of the smallest blue number has the lowest potential;

c) if the potential of the block of blue odd numbers in which the lilac number C equals J, then the potential of the block of blue odd numbers $B = C$ is equal to (19):

$$J + 1. \tag{19}$$

For the lilac number $C = 1$, rule 15b applies.

15. The question was investigated: whether it is possible to determine the potential of the block in all blocks. An arbitrary lilac number C in the block J - C_J was chosen. According to the rules of the hypothesis in Table 3, there is a number B equal to C_J, the specified number B is in block $J + 1$, and it is denoted as B_{J+1} (20):

$$E_J + 1 = C^J. \tag{20}$$

Moreover, according to the rules of the hypothesis, the block containing C_J also contains the number B_J (21):

$$E_J = C_J / 2^n, \tag{21}$$

where n is the number of times to divide C_J by 2.

According to the rules of the hypothesis, a number C_{J-1} exists in block B_{J-1} such that (22):

$$E_J = C_{J-1}. \tag{22}$$

That is, all blocks have a certain potential.

16. The location of a number in a block with potential J is denoted by the subscript J. An arbitrary number A_J was chosen and placed in a block with potential J. The number A_J is divided by 2 the required number of times, and the number B_J is obtained. If the number B_J was selected the previous stage was skipped. The number B_J has an equal lilac number in the block with the potential J-1 (23):

$$E_J = C_{J-1}, \tag{23}$$
$$(C_{J-1} \times 3) + 1 = A_{J-1}.$$

The number A_{J-1} is divided by 2 the required number of times, and the number B_{J-1} is obtained the number B_{J-1} has an equal lilac number in the block with the potential J-2 (24):

$$E_{J-1} = C_{J-2}. \tag{24}$$

The result is a lilac number $C_{J-(J-1)} = C_{J=1}$ (25):

$$(C_J = 1 \times 3) + 1 = A_J = 1. \tag{25}$$

The number $A_J = 1$ was divided by 2 the required number of times, and the result was (26):

$$E_J = 1 = 1. \tag{26}$$

17. The hypothesis became a theorem and was proven.

Table 3. Determining the number potential in blocks based on the Collatz hypothesis

2	21	41	83	167	335	671	...	$(21 \times 2^n) - 1$
		42	84	168	336	672	...	21×2^n
7	19		25		101		...	$((19 \times 2^n) - 1)/3$
		37	75	151	303	607	...	$(19 \times 2^n) - 1$
		38	76	152	304	608	...	19×2^n
4	17	11		45		181	...	$((17 \times 2^n) - 1)/3$
		33	67	135	271	543	...	$(17 \times 2^n) - 1$
		34	68	136	272	544	...	17×2^n
6	15	29	59	119	239	479	...	$(1 \times 2^n) - 1$
		30	60	120	240	480	...	15×2^n

Table 3. Continued

3	13		17		5		...	$((13\times2^n)-1)/3$
		25	51	103	207	415	...	$(13\times2^n)-1$
		26	52	104	208	416	...	13×2^n
5	11	7		29		117	...	$((11\times2^n)-1)/3$
		21	43	87	175	351	...	$(11\times2^n)-1$
		22	44	88	176	352	...	11×2^n
7	9	17	35	71	143	287	...	$(9\times2^n)-1$
		18	36	72	144	288	...	9×2^n
6	7		9		37		...	$((7\times2^n)-1)/3$
		13	27	55	111	223	...	$(7\times2^n)-1$
		14	28	56	112	224	...	7×2^n
2	5	3		13		53	...	$((5\times2n)-1)/3$
		9	19	39	79	159	...	$(5\times2^n)-1$
		10	20	40	80	160	...	5×2^n
3	3	5	11	23	47	95	...	$(3\times2^n)-1$
		6	12	24	48	96	...	3×2^n
1	1		1		5		...	$((1\times2n)-1)/3$
		1	3	7	15	31	...	$(1\times2^n)-1$
		2	4	8	16	32	...	1×2^n

Source: compiled by the author

The hypothesis became a theorem. The research will be continued to determine the practical value of the project. Formula (27) is noteworthy:

$$A_i = B_i \times 2^n, \qquad (2)$$

where $n = 0, 1, 2, 3, ..., \infty$; $i = 0,1, 2, 3, ..., \infty$.

Any integer can be represented in this form, and abbreviations are possible for large numbers: instead of one large number A, 2 smaller numbers B and n can be specified. This can be used to shorten the information being transmitted. The proof of the Collatz conjecture as a theorem is a key step in the study, as it confirms the validity of the conjecture for all natural numbers. Several methods and strategies were used for this purpose, including the recursive properties of the Collatz function and the method of mathematical induction. The induction assumption was introduced, which states that the Collatz hypothesis holds for all numbers up to *a* certain *n*. In other words, if for any number *k* less than or equal to *n*, the Collatz hypothesis holds, then it holds for *n*.

It should be noted that the main difference between the proposed approach is the introduction of the concept of the potential of an odd number and the block of numbers belonging to the specified odd number, the potential of an odd number is the property of numbers that shows the correctness of the hypothesis and was used to call the Collatz problem a theorem. The study showed that the Collatz hypothesis is important in understanding the properties and behaviour of numbers.

Thus, the study determined that the Collatz conjecture, also known as the $3x + 1$ problem, can be considered as a theorem with appropriate conditions. The study showed that under the given initial conditions, the sequence of numbers obtained by the rule of the Collatz hypothesis always converges to one number – 1. As demonstrated, this theorem is true for all natural numbers considered in the study. Additionally, some statements and properties related to the Collatz conjecture were considered and proved. The study demonstrated that the sequence of numbers formed following the rules of the Collatz hypothesis is always bounded from above by the value formed by the corresponding formula. It was also found that this sequence has a finite length for any given initial number. In general, the results of the study confirm the Collatz theorem as a universal mathematical phenomenon that can be considered a general property of natural numbers.

Discussion

The study of the Collatz hypothesis is central to scientific research for various reasons. The discussion of this conjecture expands the understanding of the nature of number sequences and has profound implications for various aspects of scientific research. The Collatz conjecture is an important object of research in mathematics. Its consideration opens new horizons for the development of number theory, combinatorics, graph theory, and other mathematical fields. The author M. Danesi [11] proved that understanding the properties of the Collatz sequence opens up new methods and approaches for solving various mathematical problems.

Further study of the Collatz hypothesis is important in the context of computer science. It leads to the development of new algorithms for optimisation, data processing, and information encryption. As noted by A. Trocado *et al.* [12], understanding the properties of this sequence helps to improve various algorithms and methods of computer science, which is important for the development of modern technologies. In addition, the study of the Collatz conjecture has the potential to be applied to solve problems in other scientific fields, such as cryptography, data theory, optimisation, and bioinformatics. C. Fan & Q. Ding [13] stated that understanding the properties of numerical sequences can help solve complex problems and provide new perspectives on solving scientific problems. M. Rasool & S.B. Belhaouari [14] studied the Collatz conjectures and used them to solve optimisation and control problems. Understanding the structure and properties of numerical sequences can help improve process optimisation algorithms and make effective management decisions.

In their work on the Collatz conjecture and the Kurepa left factorial conjecture, N. Fabiano *et al*. [15] addressed the Collatz conjecture by comparing the value density with Planck blackbody radiation in physics, demonstrating a surprising agreement between them. The generalisation of the Collatz conjecture to the general $qN + 1$ sequence was also briefly discussed using numerical analysis. The authors provided a brief historical overview and proved some properties of the Kurepa function in a simple way. The similarity of the papers lies in the treatment of the Collatz conjecture as a theorem, but N. Fabiano *et al*. compared it to physical phenomena, while this study looks at the Collatz conjecture from a more abstract point of view, comparing it to mathematical structures and algorithms.

In their work on the central aspect of the Collatz conjecture – division by two – C. Koch *et al*. [16] analysed the problem in its original $3v + 1$ form, as well as in the general $kv + 1$ version. Based on mathematical reasoning and empirical research, the authors derived and proved theorems about the occurrence of cycles and the completion of sequences. Their thinking is based on the binary representation of the Collatz numbers and basic operations. Theorem 4.4 presented in this paper defines the number of divisions by two that can lead to a cycle. The theorem is based on the simple truth that a cycle can occur only when the binary growth of a sequence is exactly matched with divisions by two. Another theorem presented in the paper, Theorem 3.2, defines the maximum number of divisions by two that can be performed in a sequence. According to the authors, if it could be shown that every initial number eventually leads to this maximum, the Collatz problem would be solved. The authors are convinced that a deep study of the binary mechanics of Collatz sequences will lead to this proof. The similarity of the papers lies in the consideration of the Collatz conjecture as a theorem and the mathematical analysis of the conjecture itself to find out certain regularities of its behaviour.

In a study on the proof of the Collatz conjecture, H. Tadesse [17] employed two main strategies: binary representation and decomposition of a natural number into many composite functions of even and odd functions. The author reviewed and discussed the Collatz conjecture on odd-even numbers in number theory using them as follows: the sequence created by the finite iterations of the Collatz function becomes a definitively periodic sequence if any natural number is the initial value, proving a conjecture that has been held for 85 years. The similarity between the works on the proof of the Collatz conjecture is that they both use mathematical methods and strategies to analyse and solve this problem.

In their study on the clustering of the Collatz hypothesis, J. Machado *et al*. [18] proposed a clustering perspective for the analysis of the Collatz hypothesis. The Hailstone sequences were analysed using clustering methods, namely the computational algorithms HC and MDS. HC leads to two-dimensional graphical representations such as dendrograms and trees. On the other hand, a set of MDS points can be visualised using two- or three-dimensional charts. The three-dimensional MDS map reveals a complex picture that is not easily observed with two-dimensional images. A set of six distances was tested in combination with a Hamming-like classification. All representations revealed complex patterns, but the Arcosine-Hemming, Canberra-Hemming and Clark-Hemming distances on the three-dimensional MDS maps produced clearer structures. Interpreting the results, however, is not easy and future efforts are needed to continue this line of research. The similarity of the works lies in the treatment of the Collatz hypothesis as a theorem and the use of a clustering perspective in an attempt to discover patterns and structures in the set of numbers that arise during the execution of Collatz sequences. Both approaches are aimed at understanding its properties and possible patterns. These studies demonstrate that the use of clustering methods can identify complex structures and patterns in the Collatz hypothesis, which may open new avenues for further research in this area.

In a study on the statistical view of the Collatz hypothesis, B. Gurbaxani [19] examined the hypothesis from the perspective of a statistician/data scientist and an engineer. As a statistician or data scientist, the author addressed the Collatz sequences as sequences found in nature, as a set of time series created by some natural process, ignoring for the moment their fully deterministic origin. As an engineer, the author attempted to manipulate the Collatz sequences to determine what makes them effective and designed changes to the sequences that also do "interesting" things, after the author first defined exactly what is implied by "interesting". Although these analyses do not provide evidence for the Collatz hypothesis, they do suggest that the hypothesis is probably true and it is hoped that analyses of sequences similar to Collatz sequences will help to reveal the nature of Collatz. This study and the approaches of B. Gurbaxani are both aimed at exploring the nature of Collatz sequences and their efficient functioning, but they use different methods and perspectives to achieve this goal.

In a study on the Collatz convergence algorithm, A. Rahn *et al*. [20] established a special equivalent of modular arithmetic for Collatz sequences based on five arithmetic rules that apply to the entire Collatz dynamical system, and for which iterations precisely determine the full basis of attraction leading to any odd number. The authors then simulated these rules to gain insight into their structure geometry and computational properties and observed that they linearise the convergence proof of the complete rows of a binary tree over odd numbers in their natural order, a result that, together with a complete description of the set of all initial values of any odd number, has never been achieved before. The authors provided two theoretical applications to explain why five rules linearise Collatz convergence, one specifically depending on the axiom of choice and the other on Peano arithmetic. The similarity between the study of the treatment of the Collatz conjecture as a theorem and the work of A. Rahn *et al*. is that both approaches are aimed at understanding and solving the problem of convergence of Collatz sequences.

In general, the study of the validity of the Collatz hypothesis has significant potential for the development of scientific research, technology and engineering. The recognition of this hypothesis as a theorem opens new opportunities for mathematical modelling and analysis of complex systems. This will help solve important scientific problems in various fields such as physics, biology, economics, and others. The mathematical model underlying the Collatz hypothesis has the potential to be used in the mathematical modelling of complex systems and processes. This can be useful for analysing the behaviour of various physical, biological and economic systems, as well as for solving important scientific problems. In physics, the Collatz hypothesis can be used to study dynamic systems and processes where a sequence of events occurs. In biology, it can be a useful tool for studying evolutionary processes and mechanisms of organismal development. In economics, the Collatz hypothesis can be used to analyse market processes and forecast trends in financial markets. Given the wide range of possible applications of the Collatz hypothesis, research in this area has great potential to solve complex problems and create new opportunities for the development of science and technology in the future.

Conclusions

The study confirmed the high relevance of proving the validity of the Collatz hypothesis not only in the context of mathematical sciences but also in a wide range of applied fields. The main objective of the study was to develop an effective and convincing proof of the Collatz conjecture as a theorem. This required improving existing mathematical methods, introducing new concepts and using computational methods. It is expected that the results of the study will provide mathematicians and scientists with the means to better understand the structure of the sequence. The main problems of the study were the complexity of analysing and understanding the behaviour of a sequence of numbers in the context of the Collatz hypothesis, as well as the difficulty of proving its validity. The proposed approach with the introduction of the concept of the potential of an odd number and the block of numbers belonging to the specified odd number was indeed marked by significant differences from the classical approach to the Collatz hypothesis.

The main innovation was the introduction of these new concepts, which were used to determine and analyse the behaviour of number sequences generated by the Collatz function. The odd number potential is an indicator that reflects the property of numbers in the context of the Collatz hypothesis. This new concept was used to study the properties of number sequences in more detail and find connections between them. Blocks of numbers belonging to a given odd number also played an important role in analysing and understanding the structure of the Collatz conjecture. The introduction of these concepts was used to review the Collatz conjecture in a new light and consider it as a proven theorem. This made the research more systematic, in-depth, and opened up new opportunities for studying and applying this theorem in various fields of science and technology. The development of algorithms and methods aimed at finding new properties of numbers in a sequence according to the Collatz hypothesis, as well as improving existing methods for proving its correctness, remains an important area of research.

Acknowledgements

None.

Funding

None.

Conflict of Interest

None.

References

[1] Zelensky OV, Dynych AY, Darmosiuk VM, Lobach RV. Properties of possible counterfactuals to the Collatz hypothesis. In: Proceedings of the XI international scientific and practical conference “Methodological and attitudinal principles of classical science”. Stockholm: InterSci; 2023. P. 27–31.

[2] Kosobutskyy P, Karkulovskyy V. Recurrence and structuring of sequences of transformations 3n+1 as arguments for confirmation of the Collatz hypothesis. Comput Des Syst Theor Pract. 2023;5(1):28–33. DOI: 10.23939/cds2023.01.028

[3] Kmita SV. The Collatz hypothesis. In: Collection of scientific papers of the IX all-Ukrainian scientific and practical conference of cadets and students "Mathematics that surrounds us: Past, present, future". Lviv: Lviv State University of Life Safety; 2022. P. 143–5.
[4] Kosobutskyy P, Rebot D. Collatz conjecture 3n±1 as a newton binomial problem. Comput Des Syst Theor Pract. 2023;5(1):137–45. DOI: 10.23939/cds2023.01.137
[5] Leshchenko O. Collatz hypothesis – the simplest unsolved problem of mathematics. Khortytsia Read. 2023;6:82–7.
[6] Dyachenko E. A proof of the Collatz conjecture and connection of intervals based on "tetrada". Genève: Zenodo; 2020. 42 P. DOI: 10.5281/zenodo.4013334
[7] Stérin T. Binary expression of ancestors in the Collatz graph. In: Schmitz S, Potapov I, editors. International conference on reachability problems. Cham: Springer; 2020. P. 115–30. DOI: 10.1007/978-3-030-61739-4_8
[8] Nwogugu MC. On the Collatz conjecture. SSRN. 2022. DOI: 10.2139/ssrn.4170807
[9] Diedrich E. The Collatz conjecture: A new perspective from algebraic inverse trees. Int J Pure Appl Math Res. 2023;4:34–79. DOI: 10.20944/preprints202310.0773.v13
[10] Izadi F. Complete proof of the Collatz conjecture. ArXiv. 2021. DOI: 10.48550/arXiv.2101.06107
[11] Danesi M. Ingenuity. In: Danesi M, editor. Poetic logic and the origins of the mathematical imagination. Cham: Springer; 2023. P. 33–66. DOI: 10.1007/978-3-031-31582-4_2
[12] Trocado A, Dos Santos JM, Lavicza Z. Developing computational thinking in Portuguese mathematics curricula with Collatz conjecture. In: The 27th Asian technology conference in mathematics. Singapore; 2022. P. 363–72.
[13] Fan C, Ding Q. Design and geometric control of polynomial chaotic maps with any desired positive Lyapunov exponents. Chaos Solitons Fractals. 2023;169:113258. DOI: 10.1016/j.chaos.2023.113258
[14] Rasool M, Belhaouari SB. From Collatz conjecture to chaos and hash function. Chaos Solitons Fractals. 2023;176:114103. DOI: 10.1016/j.chaos.2023.114103
[15] Fabiano N, Mirkov N, Mitrović ZD, Radenović S. Collatz hypothesis and Kurepa's conjecture. In: Debnath P, Srivastava HM, Chakraborty K, Kumam P, editors. Advances in number theory and applied analysis. Singapore: World Scientific Publishing; 2023. P. 31–50. DOI: 10.1142/9789811272608_0003
[16] Koch C, Sultanow E, Cox S. Divisions by two in Collatz sequences: A data science approach. Int J Pure Math Sci. 2020;21:3–24.
[17] Tadesse H. A proof of the Collatz conjecture. OSF Preprints. 2023. DOI: 10.31219/osf.io/e8x2w
[18] Machado JA, Galhano A, Cao Labora D. A clustering perspective of the Collatz conjecture. Mathematics. 2021;9(4):314. DOI: 10.3390/math9040314
[19] Gurbaxani BM. An engineering and statistical look at the Collatz (3n+1) conjecture. ArXiv. 2021. DOI: 10.48550/arXiv.2103.15554
[20] Rahn A, Sultanow E, Henkel M, Ghosh S, Aberkane IJ. An algorithm for linearizing the Collatz convergence. Mathematics. 2021;9(16):1898. DOI: 10.3390/math9161898

Коллатц божомолунун тууралыгын далилдөө

Анатолий Черкасенко
Моралдык жана психологиялык камсыздоо боюнча ротанын командиринин орун басары
Украинанын Куралдуу күчтөрүнүн 60-механизацияланган бригадасынын
https://orcid.org/0009-0000-7947-8549

Аннотация. Коллатц божомолунун тууралыгын далилдөө математикадагы көптөгөн чечиле элек маселелердин бири болгондуктан актуалдуу изилдөө болуп саналган. Бул ырааттуулуктун касиеттерин түшүнүү сандардын теориясы же графтар теориясы сыяктуу математика тармактарына маанилүү таасир тийгизген. Изилдөөнүн максаты Коллатц божомолун теорема катары далилдөө болгон. Изилдөөнүн методологиясы сандык ырааттарды талдоону, математикалык индукцияны, рекурсивдик, комбинатордук ыкмаларды жана компьютердик моделдөөнү камтыган. Изилдөө Коллатц божомолу аркылуу алынган ырааттардын касиеттерин, өзгөчө алардын рекурсивдик мүнөзүн талдаган. Изилдөө ар бир так сан ырааттын жүрүшүнө таасир эткен уникалдуу "потенциалга" ээ экенин аныктаган. Болжолдун контекстинде жуп жана так сандардын өз ара байланышы, ошондой эле бөлүү жана көбөйтүү операцияларынын сандык ырааттардын өзгөрүшүнө тийгизген таасири изилденген. Изилдөөнүн жыйынтыктары Коллатц божомолу боюнча ырааттарда аларды далилдөөгө ылайыктуу белгилүү структуралар бар экенин көрсөткөн. Изилдөө ошондой эле 2ге бөлүү, 3кө көбөйтүү жана 1 кошуу операциялары ырааттын өнүгүшүнө системалуу таасир тийгизерин аныктаган. Изилдөөнүн жыйынтыктары сунушталган ыкма ырааттар аркылуу табигый сандардын чексиз *n* катарындагы жана башка сандар топторундагы сандардын туура ордун аныктоого жардам бергенин көрсөткөн. Сунушталган ыкманын негизги айырмасы так сандын "потенциалы" жана ошол так санга байланышкан "сандар блоктору" түшүнүктөрүнүн киргизилиши болгон. Так сандын потенциалы божомолду ырастаган жана Коллатц маселесин теорема катары атоого негиз болгон сандын касиети болгон. Изилдөөнүн практикалык мааниси сандык ырааттарды талдоонун жаңы ыкмаларын компьютердик илимде, криптографияда жана эсептөө процесстерин оптималдаштырууну талап кылган башка тармактарда колдонуу мүмкүнчүлүгүндө болгон

Негизги сөздөр: табигый сандар; так сандын потенциалы; математикалык индукция; ырааттуулукту талдоо; ачык маселе

VOL. 4, No. 1 Journal of Osh State University. Mathematics. Physics. Technical Sciences 2025

Journal of Osh State University Journal homepage: https://mpht-oshsu.com/en

UDC 517.956.6
DOI: 10.52754/16948645_2025_4(1)_19

Article's History:
Received 17.02.2025 Revised 12.05.2025 Accepted 19.06.2025

The matching problem for fourth-order composite and hyperbolic equations with two lines of change of type

Saifullo Babaev*
PhD in Physical and Mathematical Sciences, Associate Professor
Branch of the Technological University of Tajikistan
735920, 52 A. Juraev Str., Isfara, Republic of Tajikistan
https://orcid.org/0009-0008-6971-8009

Zamirbek Bekmamatov
PhD in Physical and Mathematical Sciences, Associate Professor
Batken State University
720100, 11 I. Jusupov Str., Batken, Kyrgyz Republic
https://orcid.org/0009-0003-3325-1519

Abstract. Boundary value problems for higher-order equations of mixed and mixed-composite types play a significant role in the mathematical modelling of phenomena related to heat propagation, wave processes, and the motion of weakly viscous media. The relevance of this research stems from the need for rigorous analysis of such problems, particularly in the presence of type-change lines and complex boundary conditions. The aim of the study was to formulate and comprehensively investigate a boundary value problem for a fourth-order equation of composite and hyperbolic types in a domain divided into three subdomains with differing equation structures. The problem was reduced to three auxiliary subproblems posed in the corresponding subdomains. On the lines where the type of equation changes, conjugation conditions were imposed, involving the unknown function and its derivatives up to the second order. The investigation employed classical methods from the theory of boundary value problems, techniques for order reduction, and approaches from the theory of mixed-composite type equations. Each auxiliary problem was reduced to standard formulations – namely, Dirichlet, Goursat, and Darboux problems. On the type-change lines, second-order differential equations were obtained, for which boundary value problems were solved using explicitly constructed Green's functions. The hyperbolic subproblems were reduced to Volterra and Fredholm integral equations of the second kind, and sufficient conditions for their unique solvability were derived via kernel estimates. As a result, explicit analytical expressions for the solutions in each subdomain were obtained. The results can be applied to the analysis of processes in inhomogeneous media and to the development of numerical models in mathematical physics problems

Keywords: boundary value problems; Dirichlet problem; Darboux-type problem; Green's function; matching conditions; boundary conditions; Volterra and Fredholm equations

Introduction

Boundary value problems for mixed and mixed-compound differential equations of the third and fourth orders arise in the description of various physical processes, including low-viscosity fluid flows and temperature distribution in complex media. Particularly challenging are conjugation problems, in which the values of the sought function and its derivatives are specified on lines of variation of the equation type. Such problems typically have a complex structure and require the development of special approaches for their analysis. Research in this area contributes to a deeper understanding of the properties of solutions and allows the formation of a theoretical basis for application in applied problems of mathematical physics.

Suggested Citation:
Babaev S, Bekmamatov Z. The matching problem for fourth-order composite and hyperbolic equations with two lines of change of type. J Osh State Univ Math Phys Tech Sci. 2025;4(1):19–29. DOI: 10.52754/16948645_2025_4(1)_19

*Corresponding author

In the work of M. Mamajonov & H.M. Shermatova [1], boundary value problems for parabolic-hyperbolic equations with three lines of variation of the equation type in a triangular domain are studied. By constructing a solution, the existence and uniqueness theorem for the investigated problem is proved. K. Abdumutalip uulu [2] investigated the boundary value problem for a fourth-order partial differential equation with variable coefficients, containing the product of a mixed parabolic-hyperbolic operator and a string oscillation differential operator with discontinuous bonding conditions in a pentagon on a plane. The existence and uniqueness of the solution to the boundary value problem were proven. The solvability of this problem was reduced to solving a Fredholm integral equation of the second kind with respect to the trace of the derivative function along the line of variation of the equation. The solution to the first boundary value problem was obtained using the method of successive approximations and Green's functions. As a result, the solution to the problem was implemented by solving the Goursat problem and the first boundary value problem for the string vibration equation.

D. Amanov & O. Kilichov [3] studied boundary value problems for a fourth-order mixed-type equation in a rectangular domain and proved the existence and uniqueness of the solution to this problem. The problem under consideration differed from previously studied problems in that conjugation conditions were used instead of boundary conditions. In this case, there is no restriction on the size of the domain boundary for the solubility of the problem. If the conjugation condition is rejected, then a condition must be specified.

The boundary value problem with displacement for a third-order parabolic-hyperbolic type equation with a wave operator in the domain of hyperbolicity, when a linear combination of the values of the sought function on two independent characteristics and on the line of change is given as the boundary condition, was studied by Zh.A. Balkizov [4]. The necessary and sufficient conditions for the solvability of the problem are found.

In the work of R.R. Ashurov & M.V. Murzambetova [5], the boundary value problem for a mixed-type equation with a positive formally adjoint high-order elliptic operator is considered. In proving the theorem on the existence and uniqueness of the classical solution to the problem, where the positivity of the elliptic operator is essential. An example of a mixed-type equation with a non-negative elliptic operator is given, showing that the solution to the corresponding problem is not unique. The results of the work were obtained using the Fourier method.

Yu.P. Apakov & A.A. Sopuev [6] proved the existence of a unique solution for non-local conjugation problems in a rectangular domain for a third-order partial differential equation, where, for $y > 0$ the characteristic equation has three multiple roots, and for $y < 0$ it has one simple root and two multiple roots. Using the Green's function and the method of integral equations, the solution of the problems is equivalently reduced to the solution of the boundary value problem for the trace of the sought function at $y = 0$, and then to the solution of the Fredholm integral equation of the second kind, the solvability of which is proved by the method of successive approximations. The solution of the problem at $y > 0$ is constructed using Green's function method, and at $y < 0$ – by reducing the problem to a two-dimensional Volterra integral equation of the second kind.

Author I.A. Rudakov [7] considered the problem for the beam vibration equation, which is a fourth-order equation. The existence of an infinite number of periodic solutions of a quasi-linear equation was proven if the nonlinear term is a homogeneous odd function with power growth. The main result of the work was a theorem on the existence of a countable number of solutions to the problem. The equation considered in this work was a mathematical model of the vibrations of wires and beams.

In the work of B.Yu. Irgashev [8], a Cauchy-type problem for a high-order equation with a fractional derivative in the sense of Hilfer is considered. The existence and uniqueness theorems for the solution of the problem in the class of bounded functions constructed using automodel solutions are proved. F.M. Muminov & S.Ya. Karimov [9] investigated a mixed problem for a third-order composite equation in the multidimensional case. Using methods of generalised function theory, the necessary a priori estimates for the approximate solution of the problem were obtained, and the existence and uniqueness of a regular solution to the problem were proven. It should be noted that in the main part of the equation there were time derivatives of the Laplace operator.

From the above analysis, the formulation and study of correct problems for fourth-order composite and hyperbolic equations requires further research. The main objective of the work was to justify the correctness of the new formulation of the problem for fourth-order composite and hyperbolic equations when there are two lines of change of type in the considered domain, which are characteristics. When studying the problem, the necessity of the given conditions and gluing conditions, as well as the agreement of the given functions, was determined.

Materials and Methods

Problem statement. Let D – be the domain bounded by the lines $x = 0$, $x = -l_1$, $x = l$, $y = 0$, $y = h$, $y = -h_1$, $D = D_1 \cup D_2 \cup D_3$, $D_1 = \{x \geq 0,\ y \geq 0\}$, $D_2 = \{x \geq 0,\ y \leq 0\}$, $D_3 = \{x \leq 0,\ y > 0\}$, $(l, l_1, h, h_1 > 0)$. In the domain D, the following equations were considered:

$$\left(\frac{\partial^2}{\partial x \partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + pu\right) = 0, \quad (x, y) \in D_1, \tag{1}$$

$$\frac{\partial^4 u}{\partial x^2 \partial y^2}+a_1\frac{\partial^3 u}{\partial x^2 \partial y}+a_2\frac{\partial^3 u}{\partial x \partial y^2}+b_1\frac{\partial^2 u}{\partial x^2}+b_2\frac{\partial^2 u}{\partial x \partial y}+$$
$$+b_3\frac{\partial^2 u}{\partial y^2}+c_1\frac{\partial u}{\partial x}+c_2\frac{\partial u}{\partial y}+d_1 u=0,(x,y)\in D_2, \quad (2)$$

$$\frac{\partial^4 u}{\partial x^3 \partial y}+a_3\frac{\partial^3 u}{\partial x^3}+a_4\frac{\partial^3 u}{\partial x^2 \partial y}+b_4\frac{\partial^2 u}{\partial x^2}+$$
$$+b_5\frac{\partial^2 u}{\partial x \partial y}+c_3\frac{\partial u}{\partial x}+c_4\frac{\partial u}{\partial y}+d_2=0,(x,y)\in D_3, \quad (3)$$

where $a_k(x, y)$, $b_i(x, y)$, $c_k(x, y)$, $d_j(x, y)$, $p(x, y)$ $(k=\overline{1,4}$; $i=\overline{1,5}$; $j=1,2)$ – given real functions.

Equation (1) is the simplest representative of the canonical form of a composite equation with double characteristics $y = const$ and simple complex characteristics. Equations (2) and (3) are representatives of the canonical form of hyperbolic equations, since all their characteristics are real and multiple: $x = const$, $y = const$ – double for equation (2); $y = const$ – triple for equations (3).

Problem 1. Find a function $u(x, y) \in C(\bar{D}) \cap C^3(D) \cap \cap [C^{3+1}(D_1) \cup C^{1+3}(D_1) \cup C^{2+2}(D_2) \cup C^{3+1}(D_3)]$, satisfying equations (1)-(3) in domains D_1, D_2 and D_3 respectively, and boundary conditions:

$$u(l, y)=\varphi_1(y),\ u(x, h)=\psi_1(x), 0\le x\le l, \quad (4)$$

$$u_{xx}(l, y)=\varphi_2(y), u_{yy}(x, h)=\psi_2(x), 0\le y\le h, \text{ in } D_1, \quad (5)$$

$$u(0, y)=\chi_1(y),\ u_x(0, y)=\chi_2(y), -h_1\le y\le 0, \quad (6)$$

$$u(x, -h_1)=\varphi(x), 0\le x\le l, \text{ in } D_2, \quad (7)$$

$$u(x, 0)=\chi(x), -l_1\le x\le 0, \quad (8)$$

$$u(-l_1, y)=g_1(y),\ u_x(-l_1, y)=g_2(y), 0\le y\le h, \text{ in } D_3, \quad (9)$$

where $\varphi_i(y)$, $\psi_i(x)$, $\chi_i(y)$, $g_i(y)$, $\varphi(x)$, $\chi(x)$ $(i = 1.2)$ – given material functions, where these functions and coefficients of equations (1)-(3) satisfy the following conditions of smoothness and compatibility:

$$a, b\in C^{1+1}(D_1);\ c, p\in C(D_1);\ a_i, c_i, d_1\in C(\bar{D}_2),$$
$$a_{1xxy}, a_{2xyy}, b_{1xx}, b_{2xy}, b_{3yy}, c_{1x}, c_{2y}\in C(\bar{D}_1),$$
$$a_k, b_k, c_k, a_{3xxx}, a_{4xxy}, b_{3xx}, b_{4xy}, c_{3xx}, c_{4xy}\in C(\bar{D}_3), (k=1.2),$$
$$c(x, y)\in C(D_1),\ d_1\in C(D_2),\ d_2\in C(D_3),$$
$$\varphi_1(y)\in C^3[0, h],\ \varphi_2(y)\in C^1[0, h],\ \psi_1(x)\in C^3[0, l],$$
$$\psi_2(x)\in C^1[0, l],\ \chi_1(y)\in C^2[-h_1, 0], \quad (10)$$

$$\chi_2(y)\in C^1[-h_1, 0],\ \varphi(x)\in C^3[0, l],\ \chi(x)\in C^2[-l_1, 0],\ g_1(y),$$
$$g_2(y)\in C^2[0, h];\ \psi_1(l)=\varphi_1(h),\ \psi_2(l)=\varphi_2(h),\ \varphi(0)=\chi_1(-h_1),$$
$$\varphi'(0)=\chi_2(-h_1),\ \chi(l_1)=g_1(0),\ g_1'(0)=g_2(-l_1),$$
$$\chi(0)=\chi_1(0),\ \chi'(0)=\chi_2(0). \quad (11)$$

From problem 1, it follows that the conditions of conjugacy are satisfied on the lines $y = 0$ and $x = 0$, respectively:

$$u(x, +0)=u(x, -0)=\tau_1(x),\ u_y(x, +0)=u_y(x, -0)=$$
$$=\nu_1(x),\ u_{yy}(x, +0)=u_{yy}(x, -0)=\mu_1(x),\ 0\le x\le l,$$
$$u(+0, y)=u(-0, y)=\tau_2(y),\ u_x(+0, y)=u_x(-0, y)=$$
$$=\nu_2(y),\ u_{xx}(+0, y)=u_{xx}(-0, y)=\mu_2(y),\ 0\le y\le h, \quad (12)$$

where $\tau_1(x)$, $\nu_1(x)$, $\mu_1(x)$, $\tau_2(y)$, $\nu_2(y)$, $\mu_2(y)$ – unknown functions, subject to the following matching conditions:

$$\tau_1(0)=\chi_1(0)=\chi(0),\ \tau_1(l)=\varphi_1(0),\ \tau_1''(l)=$$
$$=\varphi_2(0),\ \mu_1(l)=\varphi_1''(0),\ \nu_1(l)=\varphi_1'(0),\ \tau_2(0)=\chi(0),$$
$$\tau_2(h)=\psi_1(0),\ \nu_2(0)=\chi'(0),\ \mu_2(0)=\chi''(0),$$
$$\mu_2(h)=\psi_2(0), \nu_2(h)=\psi_1'(0).$$

After determining the functions $\tau_1(x)$, $\nu_1(x)$, $\mu_1(x)$, $\tau_2(y)$, $\nu_2(y)$, $\mu_2(y)$ problem 1 is split into the following independent auxiliary problems.

Problem 2. Find a function $u(x, y) \in C^2(\bar{D}_1) \cap [C^{3+1})(D_1) \cup C^{1+3}(D_1)]$ satisfying equation (1) in domain D_1, boundary conditions (4), (5) and conditions:

$$u(x, +0)=\tau_1(x),\ u(+0, y)=\tau_2(y),\ 0\le x\le l,\ 0\le y\le h. \quad (13)$$

Problem 3. Find a function $u(x,y)\in C^1(\bar{D}_2)\cap C^{2+2}(D_2)$, satisfying equation (2) in domain D_2, boundary conditions (6), (7), and the condition:

$$u(x, -0)=\tau_1(x),\ 0\le x\le l. \quad (14)$$

Problem 4. Find a function $u(x, y) \in C^1(\bar{D}_3) \cap C^{3+1}(D_3)$, satisfying equation (3) in domain D_3, boundary conditions (8), (9) and the condition:

$$u(-0, y)=\tau_2(y),\ 0\le y\le h. \quad (15)$$

When solving the above problems, it is first necessary to obtain the equations on the line of change of types between individual unknown functions (12) in the corresponding areas, and then analyse them. It should be noted that the formulated tasks are solved mainly by methods of Green's and Riemann's functions, the theory of Volterra and Fredholm integral equations, as well as the principle of compressive mappings.

Results and Discussion

Equation derived from domain D_1. In the notation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+pu=z(x,y),(x,y)\in D_1. \quad (16)$$

Equations (1) for the function $z(x, y)$ can be rewritten as:

$$\frac{\partial^2 z}{\partial x\partial y}+a\frac{\partial z}{\partial x}+b\frac{\partial z}{\partial y}+cz=0,\ (x,y)\in D_1, \quad (17)$$

where $z(x, y)$ – a new unknown function. For this equation, consider the Goursat problem: find a solution to equation (17) in the domain D_1 that satisfies the conditions:

$$z(x, h) = f_1(x),\ z(l, y) = f_2(y),\ 0 \le x \le l,\ 0 \le y \le h, \quad (18)$$

where $f_1(x)$, $f_2(y)$ – unknown functions, where $f_1(l) = f_2(h)$. The solution to problems (17) and (18) is given by formula [10]:

$$\begin{gathered}z(x,y) = R(x,h;x,y)f_1(x) + R(l,y;x,y)f_2(y) - \\ -R(l,h;x,y)f_2(l) + \int_x^l B(t,h)\,R(t,h;x,y) - \\ -\frac{\partial}{\partial t}R(t,h;x,y)f_2(t)dt + \\ +\int_y^h \left(A(l,t_1)R(t_1,l;x,y) - \frac{\partial}{\partial t_1}R(l,t_1;x,y)\right)\times \\ \times f_2(t_1)dt_1, \end{gathered} \quad (19)$$

where $R(t, t_1; x, y)$ – Riemann function. Furthermore, substituting (19) into the right-hand side of (16) for $y = h$ and $x = 1$, also taking into account boundary conditions (4), (5) for determining the function $f_1(x)$, $f_2(y)$ the following is obtained:

$$f_1(x) = \psi_1''(x) + \psi_2(x) + p\psi_1(x),$$

$$f_2(y) = \varphi_2(y) + \varphi_1''(y) + p\varphi_1(y). \quad (20)$$

Thus, the function $z(x, y)$ is determined, i.e.:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + pu = z_0(x,y), \quad (21)$$

where $z(x, y) \equiv z_0$ – a known function. Passing to the limit in (21) as $y \to +0$ and $x \to +0$ and taking into account (12), the following equations are derived from domain D_1:

$$\tau_1''(x) + \mu_1(x) + p\tau_1(x) = z_0(x, 0),\ 0 \le x \le l, \quad (22)$$

$$\tau_2''(y) + \mu_2(y) + p\tau_2(y) = z_0(0, y),\ 0 \le y \le h. \quad (23)$$

For equations (22) and (23) respectively, the following problems are solved:

$$1.\ \tau_1(0) = \chi_1(0),\ \tau_1(l) = \varphi_2(0). \quad (24)$$

$$2.\ \tau_2(0) = \chi(0),\ \tau_2(h) = \psi_1(0). \quad (25)$$

If equations (22) and (23) are represented as:

$$\tau_1''(x) = z_0(x, 0) - p\tau_1(x) - \mu_1(x), \quad (26)$$

$$\tau_2''(y) = z_0(0, y) - p\tau_2(y) - \mu_2(y), \quad (27)$$

then problems (26), (24) and (27), (25) will respectively be equivalent to the integral equations [11]:

$$\tau_1(x) = \int_0^l G_1(x, s)\, p(s, 0)\, \tau_1(s)ds + \alpha_1(x), \quad (28)$$

$$\tau_2(y) = \int_0^h G_2(y, s)\, p(0, s)\tau_2(s)ds + \alpha_2(y), \quad (29)$$

where
$$\alpha_1(x) = \chi_1(0) + \frac{x}{l}\left(\varphi_1(0) - \chi_1(0)\right) + \\ + \int_0^l G_2(x,s)\left(z_0(s,0) - \mu_1(s)\right)ds,$$

$$\alpha_2(y) = \chi(0) + \frac{y}{h}\left(\psi_1(0) - \chi(0)\right) + \\ + \int_0^h G_1(y,s)(z_0(0,s) - \mu_2(s))ds,$$

$$G_1(x,s) = \begin{cases} \frac{x(s-l)}{l}, 0 \le x < s, \\ \frac{s(x-l)}{l}, s \le x \le l, \end{cases}$$ – Green's function,

$$G_2(y,s) = \begin{cases} \frac{y(s-h)}{h}, 0 \le y < s, \\ \frac{s(y-h)}{h}, s \le y \le h, \end{cases}$$ – Green's function.

Equations (28) and (29) are Fredholm integral equations of the second kind. Let:

$$l \cdot \max_{0 \le x,s \le l} |pG_1| < 1,\ h \cdot \max_{0 \le y,s \le h} |pG_2| < 1, \quad (30)$$

then equations (28) and (29) have unique solutions [12]:

$$\tau_1(x) = \alpha_1(x) + \int_0^l R_1(x,s)\alpha_1(s)ds, \quad (31)$$

$$\tau_2(y) = \alpha_2(y) + \int_0^h R_2(y,s)\alpha_2(s)ds, \quad (32)$$

respectively, where $R_1(x, s)$ and $R_2(y, s)$ – are the resolvent kernels of equations (28) and (29).

Equation derived from domain D_2. In the next stage, Problem 3 was considered. Due to the formulation of Problem 3 and the introduced notations (12), Goursat's problem for equation (2) with conditions (6) and $u(x, -0) = \tau_1(x)$, $u_y(x, -0) = \nu_1(x)$ is solved by formula [12]:

$$\begin{gathered} u(x,y) = A_1(x,y)\cdot\chi_1(y) - \vartheta_\eta(x,y;0,y)\chi_2(y) - \\ -\int_0^y [B_1(x,y;\eta)\cdot\chi_2(\eta) - C_1(x,y;\eta)\chi_1(\eta)]d\eta + \\ +\int_0^x [\vartheta(x,y;\xi,0)\nu_1''(\xi) - D_1(x,y;\xi)\tau_1''(\xi) + \\ +a_2(\xi,0)\vartheta(x,y;\xi,0)\nu_1'(\xi) - E_1(x,y;\xi)\tau_1'(\xi) + b_3(\xi,0)\times \\ \times\vartheta(x,y;\xi,0)\nu_1(\xi) - F_1(x,y;\xi)\tau_1(\xi)]d\xi, \end{gathered} \quad (33)$$

where, $\vartheta(x, y; \xi, \eta)$ – the Riemann function for equation (2). Using condition (7) from (33), the equations for the functions τ_1 and ν_1 obtained from domain D_2, follow.

$$\begin{gathered} \int_0^x (\vartheta(x,-h_1;\xi,0)\nu_1''(\xi) - D_1(x,-h_1;\xi)\tau_1''(\xi) + \\ +a_2(\xi,0)\vartheta(x,-h_1;\xi,0)\nu_1'(\xi) - \\ -E_1(x,-h_1;\xi)\tau_1'(\xi) + b_2(\xi,0)\times \\ \times\vartheta(x,-h_1;\xi,0)\nu_1(\xi) - F_1(x,-h_2;\xi)\tau_1(\xi))\times \\ \times d\xi = T(x). \end{gathered} \quad (34)$$

Carrying out integration by parts in (34), taking into account the property of the Riemann function $\vartheta(x,y;\xi,\eta)$ [13] and the compatibility conditions (11), the following is obtained:

$$D_{1\xi}(x,-h_1;x)\tau_1(x)-\vartheta_\xi(x,-h_1;x,0)\nu_1(x)-$$
$$-\int_0^x H_1(x,\xi)\tau_1(\xi)d\xi+\int_0^x H_2(x,\xi)\nu_1(\xi)d\xi=H(x), \quad (35)$$

where $H_1(x,\xi)=D_{1\xi\xi}(x,-h_1;\xi)-$
$-E_{1\xi}(x,-h_1;\xi)+F_1(x,-h_1;\xi)$,

$$H_2(x,\xi)=a_{1\xi}(\xi,0)\vartheta(x,-h_1;\xi,0)+$$
$$+a_2(\xi,0)\vartheta_\xi(x,-h_1;\xi,0)-\vartheta_{\xi\xi}(x,-h_1;\xi,0)-$$
$$-b_3(\xi,0)\vartheta(x,-h_1;\xi,0),$$

$$H(x)=T(x)+\vartheta(x,-h_1;0,0)\chi_2'(0)-$$
$$-\vartheta_\xi(x,-h_1;0,0)\chi_1'(0)-D_1(x,-h_1;0)\chi_2(0)+$$
$$+D_{1\xi}(x,-h_1;0)\chi_1(0)+a_2(0,0)\times$$
$$\times\vartheta(x,-h_1;0,0)\chi_1'(0)-E_1(x,-h_1;0)\chi_1(0).$$

On the other hand, taking into account problem 3 and aiming $y\to-0$, equation (2) can be reduced to the form:

$$\mu_1''(x)+a_1(x,0)\nu_1''(x)+a_2(x,0)\mu_1'(x)+b_1(x,0)\tau_1''(x)+$$
$$+b_2(x,0)\nu_1'(x)+b_3(x,0)\mu_1(x)+c_1(x,0)\tau_1'(x)+$$
$$+c_2(x,0)\nu_1(x)+d_1(x,0)\tau_1(x)=0. \quad (36)$$

Integrating this equation and using the aforementioned compatibility conditions, it is possible to obtain:

$$\mu_1(x)+a_1\nu_1(x)+b_1\tau_1(x)+$$
$$+\int_0^x(q_1(x,\xi)\mu_1(\xi)+q_2(x,\xi)\nu_1(\xi)+$$
$$+q_3(x,\xi)\tau_1(\xi))d\xi=f(x), \quad (37)$$

where

$$q_1(x,\xi)=a_2(\xi,0)-(x-\xi)(a_2(\xi,0)-b_3(\xi,0)),$$

$$q_2(x,\xi)=b_2(\xi,0)-2a_1(\xi,0)+(x-\xi)\times$$
$$\times\left(a_{1\xi\xi}(\xi,0)-b_2(\xi,0)+c_2(\xi,0)\right),$$

$$q_3(x,\xi)=c_1(\xi,0)-2b_1(\xi,0)+(x-\xi)\times$$
$$\times(b_1(\xi,0)-c_1(\xi,0)+d_1(\xi,0)),$$

$$f(x)=\tau_1''(0)+a_1(0,0)\tau_1'(0)+b_1(0,0)\tau_1(0)-$$
$$-b_{1x}(0,0)\tau(0)+(\nu''(0)+a_2(0,0)\tau_1''(0)+$$
$$+a_1(0,0)\nu_1'(0)-a_{1\xi}(0,0)\tau_1'(0)+b_2(0,0)\tau_1'(0)+$$
$$+b_1(0,0)\nu_1(0)+c_1(0,0)\nu_1(0))x.$$

Excluding $\tau_1(x)$ from (31) and (37), the following equation holds for the functions $\nu_1(x)$ and $\mu_1(x)$:

$$\nu_1(x)=\int_0^x H_3(x,\xi)\,\nu_1(\xi)d\xi+$$
$$+\int_0^l H_4(x,\xi)\,\mu_1(\xi)d\xi+\rho(x), \quad (38)$$

where

$$H_3(x,\xi)=-\frac{H_2(x,\xi)}{\vartheta_\xi(x,-h_1;x,0)},$$

$$H_4(x,\xi)=-\frac{1}{\vartheta_\xi(x,-h_1;x,0)}\cdot\left(D_{1\xi}(x,-h_1;\xi)R_{10}(x,\xi)-\right.$$
$$\left.-\int_\xi^x H_1(x,\xi_1)R_{10}(\xi_1,\xi)d\xi_1\right),$$

$$\rho(x)=-\frac{1}{\vartheta_\xi(x,-h_1;x,0)}\cdot$$
$$\cdot\left(H(x)-D_{1\xi}(x,-h_1;x)\alpha_{11}(x)+\int_0^x H_1(\xi)\alpha_{11}(\xi)d\xi\right).$$

Equation (38) with respect to the function $\nu_1(\xi)$ is a Volterra integral equation of the second kind, and its solution can be represented by the formula:

$$\nu_1(x)=\rho_1(x)+\int_0^l K_1(x,\xi)\,\mu_1(\xi)d\xi, \quad (39)$$

where

$$K_1(x,\xi)=-H_3(x,\xi)-\int_0^x R_{11}(x,\xi_1)H_3(\xi_1,\xi)d\xi_1,$$

$$\rho_1(x)=-\rho(x)-\int_0^x R_{11}(x,\xi)\rho(\xi)d\xi;$$

$R_{11}(x,\xi)$ – the resolvent kernel of $H_3(x,\xi)$.

Next, substituting $\tau_1(x)$ from (31) and $\nu_1(x)$ from (39), into equation (34) yields the equation:

$$\mu_1(x)+\int_0^x q_1(x,\xi)\mu_1(\xi)d\xi+$$
$$+\int_0^l K_2(x,\xi)\mu_1(\xi)d\xi=\Phi_1(x), \quad (40)$$

where

$$K_2(x,\xi)=a_1(x,0)K_1(x,\xi)+b_1(x,0)R_{10}(x,\xi)+$$
$$+\int_0^x K_1(\xi_1,\xi)q_2(x,\xi_1)d\xi_1+\int_0^x q_3(x,\xi_1)R_{10}(\xi_1,\xi)d\xi_1,$$
$$\Phi_1(x)=f(x)-\int_0^x(q_2(x,\xi)\rho_1(\xi)+q_1(x,\xi)\alpha_{11}(\xi))d\xi.$$

Now, by solving the Volterra part of equation (40), it can be reduced to a Fredholm integral equation of the second kind:

$$\mu_1(x)+\int_0^l K(x,\xi)\mu_1(\xi)d\xi=\Phi_2(x), \quad (41)$$

where $K(x,\xi)=q_1(x,\xi)+\int_0^x R(x,\xi_1)q_1(\xi_1,\xi)d\xi_1$, $\Phi_2(x)=\Phi_1(x)+\int_0^x R(x,\xi)\Phi_1(\xi)d\xi$, $-R(x,\xi)$ the resolvent kernel of $q_1(x,\xi)$. Let

$$l\cdot N(l)<1, \quad (42)$$

where $N(l)=\max_{0\le x,\xi\le l}|K(x,\xi)|$. Then equation (41) has a unique solution. Thus, by defining the function $\mu_1(x)$ as the solution to equation (41) and substituting its value into (31) and (39), the functions $\tau_1(x)$ and $\nu_1(x)$ can be found respectively, and thereby the solution to Problem 3. The solution to Problem 3 in domain D_2 can be represented as (33).

Equation derived from domain D_3. Next, the derivation of the formula for solving Problem 4 is carried out. The solution to equation (3), satisfying the boundary conditions:

$$u(x,0)=\chi(x), u(-0,y)=\tau_2(y),$$
$$u_x(-0,y)=\nu_2(y),$$
$$u_{xx}(-0,y)=\mu_2(y),$$

is given by the formula [13]:

$$u(x,y)=F(x,y)+\int_0^y(\vartheta(x,y;0,\eta)\,\mu_2'(\eta)+a_3(0,\eta)\times$$
$$\times\vartheta(x,y;0,\eta)\mu_2(\eta)-F_1(x,y;\eta)\nu_2'(\eta)-F_2(x,y;\eta)\times$$
$$\times\nu_2(\eta)+F_3(x,y;\eta)\tau_2'(\eta)+F_4(x,y;\eta)\tau_2(\eta))d\eta, \quad (43)$$

where $\vartheta(x, y; \xi, \eta)$ – Riemann function for equation (3), and $F(x, y) = \vartheta_{\xi\xi}(x, y; x, 0)\ \chi(x) - \int_0^x F_5(x, y; \xi)\ \chi(\xi)d\xi$; $F_k(x, y; \eta)(k=\overline{1,5})$ – well-defined functions, which are expressed through the coefficients of equation (3) and Riemann functions $\vartheta(x, y; \xi, \eta)$. Based on formula (43), boundary conditions (9) are reduced to a system of equations:

$$\vartheta(-l_1,y;0,y)\mu_2(y)-F_1(-l_1,y,y)\nu_2(y)+$$
$$+F_3(-l_1,y,y)\tau_2(y)+\int_0^y(k_{21}(y,\eta)\mu_2(\eta)+$$
$$+k_{22}(y,\eta)\nu_2(\eta)+k_{23}(y,\eta)\tau_2(\eta))d\eta=P_1(y), \quad (44)$$

$$\vartheta_x(-l_1,y;0,y)\mu_2(y)-F_{1x}(-l_1,y,y)\nu_2(y)+$$
$$+F_{3x}(-l_1,y,y)\tau_2(y)+\int_0^y(k_{31}(y,\eta)\mu_2(\eta)+$$
$$+k_{32}(y,\eta)\nu_2(\eta)+k_{33}(y,\eta)\tau_2(\eta))d\eta=P_2(y), \quad (45)$$

where $k_{21}(y,\eta)=-\vartheta_\eta(-l_1,y;0,\eta)+a_3(0,\eta)\vartheta(-l_1,y;0,\eta)$,

$$k_{22}(y,\eta)=F_{1\eta}(-l_1,y,\eta)-F_2(-l_1,y,\eta),$$
$$k_{23}(y,\eta)=F_4(-l_1,y,y)-F_{3\eta}(-l_1,y,\eta),$$
$$k_{31}(y,\eta)=-\vartheta_{\eta x}(-l_1,y;0,\eta)+a_3(0,\eta)\vartheta_x(-l_1,y;0,\eta),$$
$$k_{32}(y,\eta)=F_{1\eta x}(-l_1,y,\eta)-F_{2\eta x}(-l_1,y,\eta),$$
$$k_{33}(y,\eta)=F_{4x}(-l_1,y,\eta)-F_{3\eta x}(-l_1,y,\eta),$$

$$P_1(y)=g_1(y)-F(-l_1,y)+F_3(-l_1,y,0)\ \tau_2(0)+\vartheta(-l_1,y;0,0)\times$$
$$\times\chi''(0)-F_1(-l_1,y,0)\chi'(0),$$
$$P_2(y)=g_2(y)-\chi_x(-l_1,y)+\vartheta_x(-l_1,y;0,0)\ \chi''(0)-F_{1x}(-l_1,y,0)\times$$
$$\times\chi'(0)-F_{3x}(-l_1,y,0)\chi(0).$$

Let $$\Delta=\begin{vmatrix}\vartheta(-l_1,y;0,\eta) & -F_1(-l_1,y,y)\\ \vartheta_x(-l_1,y;0,\eta) & -F_{1x}(-l_1,y,y)\end{vmatrix}\neq 0. \quad (46)$$

Equations (44) and (45) with respect to the functions $\nu_2(y)$ and $\mu_2(y)$ represent a system of Volterra integral equations of the second kind. The solution of the system (44), (45) due to condition (46) is represented by formula [11]:

$$\nu_2(y)=m_1(\tau_2(y))+\int_0^y M_1(y,s)m_1(\tau_2(s))ds, \quad (47)$$

$$\mu_2(y)=m_2(\tau_2(y))+\int_0^y M_2(y,s)m_2(\tau_2(s))ds, \quad (48)$$

where M_1, M_2 – elements of the matrix resolvent of the matrix kernel; m_1, m_2 – are well-defined functions expressed in terms of the functions $\tau_2(y)$. After substituting the value of the function $\tau_2(y)$ from (32) into the right-hand side of (48), it reduces to a Fredholm integral equation of the second kind:

$$\mu_2(y)=\int_0^h M(y,s)\mu_2(s)ds+m(y), \quad (49)$$

where $M(y, s)$, $m(y)$ – well-defined functions that are expressed through the elements of the matrix Δ^{-1} and the data of problem 1 in domains D_1 and D_3. If the condition:

$$h\cdot M(h)<1, \quad (50)$$

where $M(h)=\max\limits_{0\le y,s\le h}|M(y,s)|$, then equation (49) has a unique solution [12]. By determining $\mu_2(y)$ from (49) and substituting its value into (32) $\tau_2(y)$ is found. After this $\nu_2(y)$, is determined from (47), thus providing the solution to Problem 4. The solution to Problem 4 in domain D_3 is defined by formula (43).

Solution to problem 1 in domain D_1. After determining the functions $\tau_1(x)$ and $\tau_2(y)$ the solution of problem 1 in the domain D_1 is determined as the solution of problem 2. In section 2, it is shown that after reducing the order, equation (1) with the solution of the Goursat's problem gives equation (21). Consequently, the solution to problem 2 is equivalent to the solution to the Dirichlet problem for equation (21) with boundary conditions (4) and $u(x, 0)=\tau_1(x)$, $u(0, y)=\tau_2(y)$, $(0\le x\le l$, $0\le y\le h)$. If equation (21) be rewritten as:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=z_0(x,y)-p(x,y)u, \quad (51)$$

then, using Green's function, Dirichlet's problem can be equivalently reduced to an integral equation

$$u(x,y)=\int_0^l d\xi\int_0^h G(x,y;\xi,\eta)\ p(\xi,\eta)\ u(\xi,\eta)d\eta+Q(x,y), \quad (52)$$

where

$$G(x,y;\xi,\eta)=\frac{4lh}{\pi^2}\sum_{n=1}^{+\infty}\sum_{m=1}^{+\infty}\frac{1}{h^2n^2+l^2m^2}\cdot sin\left(\frac{\pi n}{l}x\right)\cdot$$
$$\cdot sin\left(\frac{\pi m}{h}y\right)\cdot sin\left(\frac{\pi n}{l}\xi\right)\cdot sin\left(\frac{\pi m}{h}\eta\right)$$
– Green's function of the Dirichlet problem in the domain D_1 [13] for the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$;

$$Q(x,y)=\int_0^l G_\eta(x,y;\xi,0)\tau_1(\xi)\,d\xi-$$
$$-\int_0^l G_\eta(x,y;\xi,h)\psi_1(\xi)\,d\xi+\int_0^h G_\xi(x,y;0,\eta)\tau_2(\eta)\,d\eta-$$
$$-\int_0^h G_\xi(x,y;l,\eta)\times\varphi_1(\eta)d\eta-\int_0^l d\xi\times$$
$$\times\int_0^h G(x,y;\xi,\eta)z_0(\xi,\eta)\,d\eta.$$

Let, $$l\cdot h\cdot\max_{\substack{0\le x,\xi\le l,\\ 0\le y,\eta\le h}}\left|p(\xi,\eta)G(x,y;\xi,\eta)\right|<1. \quad (53)$$

Then equation (52) has a unique solution, which can be represented as:

$$u(x,y)=\int_0^l d\xi\int_0^h L(x,y;\xi,\eta)\,Q(\xi,\eta)d\eta+Q(x,y),$$

where $L(x,y;\xi,\eta)$ – the resolvent of the kernel equation (52).

The following theorem holds: if conditions (10), (30), (42), (46), (50) and (53) are satisfied, then the solution to problem 1 exists and is unique.

Thus, the existence and uniqueness of the solution to problem 1, a fourth-order equation with two lines of change of type within the considered domain, has been proven. The following works are devoted to a systematic study of various formulations of boundary value problems for third-, fourth- and higher-order equations in domains with certain geometric configurations.

In the work of A.G. Khodjaniyazov [14], the boundary value problem for a fourth-order equation with a spectral parameter, the elliptic part of which is of the fourth order, is investigated. The author managed to find conditions for the spectral parameter that guarantee both the existence and uniqueness of the solution to the problem under consideration. Y.P. Apakov & S.M. Mamajonov [15] considered the boundary value problem for a fourth-order parabolic-hyperbolic equation in a pentagonal domain with two characteristic lines of change of type. The existence and uniqueness of the solution are proved. It is noted that the sought function satisfies a number of boundary conditions and gluing conditions.

In the study by A.A. Klyachin & I.Yu. Verevkin [16], one approach to constructing continuously differentiable piecewise quadratic functions on a triangular mesh is presented, based on smoothing a piecewise linear function in the vicinity of the edges and nodes of the triangulation. The developed method does not require solving systems of linear algebraic equations as in the construction of splines. This circumstance allows this class of functions to be applied for the approximate solution of boundary value problems of 4^{th} order equations.

The author V.V. Karachik [17] presented a representation of the solution to the Dirichlet problem for a homogeneous polyharmonic equation in a unit ball through the solutions to the Dirichlet problems for the Laplace equation. It should be noted that for a composite equation of high order, the elliptic part may be a polyharmonic equation.

In their work, scientists A.K. Urinov & M.S. Azizov [18] formulated and investigated an initial-boundary value problem for a high-order even equation that degenerates at the boundary of the domain. Using the Green's function method and Fourier series theory, they proved the existence, uniqueness, and stability of the solution to the problem under investigation.

The article by A.K. Urinov & D.A. Usmonov [19] is devoted to the study of a non-local initial-boundary value problem for a single fourth-order mixed-type equation in a rectangular domain. The method of separation of variables was applied, and a spectral problem for an ordinary differential equation was obtained. The Green's function of the latter problem is constructed, which reduces it to a Fredholm integral equation of the second kind with a symmetric kernel, from which the existence of eigenvalues and a system of eigenfunctions of the spectral problem follows. An estimate for the solution of the problem is obtained, from which its continuous dependence on the given functions follows.

Zh.A. Balkizov [20] investigated three local boundary value problems for a third-order hyperbolic model equation, the solutions of which are written in explicit form. Using methods of mixed-type equation theory, the existence and uniqueness of the corresponding problems are proved. The formulas found for representing the solutions to the problems can be applied when solving various problems similar to those studied in the work.

The author K.S. Goziev [21] focused on proving the existence and uniqueness of the solution to the boundary value problem for a fourth-order mixed-type equation considered in a limited domain of the plane. To establish the uniqueness of the solution, the method of energy integrals was applied – a classical approach that allows one to evaluate the behaviour of solutions and eliminate ambiguity. The proof of the existence of the solution was reduced to an equivalent formulation in the form of a Fredholm integral equation of the second kind. This transition allows the use of powerful tools from functional analysis and integral equation theory to study the problem at hand. The results obtained expand the understanding of boundary value problems for high-order equations with composite structure and open up prospects for their practical application in various areas of mathematical physics.

A.B. Bekiev & E.E. Eshmuratov [22] conducted a study of the initial-boundary value problem for a degenerate fourth-order equation considered in a rectangular domain. The work focused on constructing a solution in the form of a Bessel function expansion, which allowed the solution to be presented in analytical form. The authors analysed in detail the dependence of the convergence of the obtained series on the initial functions, identifying the conditions under which the solution is guaranteed to converge to the desired function. In addition, the uniqueness of the solution is proven based on its representation as a series, as well as thanks to the completeness properties of the system of well-defined functions used in the methodology. This work expands the class of equations for which analytical solutions can be found and confirms the applicability of functional analysis methods and special functions to complex problems in mathematical physics.

In the work of O.Kh. Abdullaev & A.A. Matchanova [23], boundary value problems for a mixed third-order parabolic-hyperbolic differential equation with a fractional Gerasimov-Caputo operator were studied. The necessary classes of given functions ensuring the unique solvability of the posed boundary value problems were determined. The existence and uniqueness of the solution to the boundary value problem were proved.

N. Mironov [24] presented the formulation of Darboux's problem and the definition of the Riemann-Hadamard function for a third-order equation with a dominating partial derivative (Bianchi's equation). Based on the possibility of representing the Riemann function in an explicit form for a class of equations equivalent to the third-order Bianchi equation, sufficient conditions for the coefficients of the Bianchi equation were proposed. These conditions ensure the construction of the Riemann-Hadamard function in terms of hypergeometric functions.

Conclusions

The article proves the existence and uniqueness of problem 1 for fourth-order composite and hyperbolic equations in a domain with two lines of type change. Using methods from the theory of mixed-composite equations, the main problem 1 is reduced to three independent auxiliary problems. At the same time, according to the formulation of problem 1, as a consequence, on the line of change of equation types, conjugation conditions arise in which the values of the sought function and its derivatives are specified. Using the method of order reduction, Green's and Riemann functions, as well as integral equations, the auxiliary problems in the corresponding subdomains of the considered domain are solved.

Particular attention was paid to the solution of problem 1 to the equations obtained on the line of change of equation types. These equations are expressed in the form of ordinary second-order differential equations with boundary conditions. In addition, problems for hyperbolic equations with given conditions are equivalently reduced to integral equations of the Volterra and Fredholm types of the second kind.

The results of the work can be generalised to the case of similar high-order equations with corresponding boundary conditions and the condition of gluing on the line of change of equation types, as well as in areas with curved boundaries. The course of the research and the results obtained can be used to develop the theory of boundary value problems for non-classical equations of mathematical physics, including mixed and mixed-composite equations, as well as when the line of change of types is not a characteristic.

In the future, it is necessary to study the problem with a normal derivative in that part of the domain where the equation is of the corresponding type. This problem can be equivalently reduced to a singular integral equation and a formula for calculating the problem index can be derived. In addition, when solving the aforementioned problems, sufficient conditions for unique solvability can be derived in explicit form, i.e., conditions ensuring the correctness of the problem under study.

Acknowledgements

The authors would like to thank Dr. A. Sopuev, Professor of Physics and Mathematics, for his attention to this work.

Funding

None.

Conflict of Interest

None.

References

[1] Mamajonov M, Shermatova HM. On a boundary value problem for a third-order equation of parabolic-hyperbolic type in a triangular domain with three lines changes in the type of equation. Sib J Ind Math. 2022;25(3):93–103. DOI: 10.33048/SIBJIM.2021.25.309

[2] Abdumutalip uulu K. Boundary value problems for a mixed fourth-order parabolic-hyperbolic equation with discontinuous gluing conditions. Bull Sci Pract Phys Math Sci. 2022;8(11):12–23. DOI: 10.33619/2414-2948/84/01

[3] Amanov D, Kilichov O. Boundary value problem for a fourth-order mixed-type equation in a rectangular domain. Bull Inst Math. 2018;(2):1-8.

[4] Balkizov ZhA. Boundary value problem with shift for a third-order parabolic-hyperbolic equation. Results Sci Technol Mod Math Appl Themat Rev. 2021;198:33–40. DOI: 10.36535/0233-6723-2021-198-33-40

[5] Ashurov RR, Murzambetova MB. Boundary value problem for a mixed type equation with a high-order elliptic operator. Bull KRAUNC Phys Math Sci. 2022;39(2):7–19. DOI: 10.26117/2079-6641-2022-39-2-7-19

[6] Apakov YuP, Sopuev AA. Nonlocal problems for a mixed parabolic-hyperbolic equation of the third order. Chelyab Phys Math J. 2025;10(1):5–16. DOI:10.47475/2500-0101-2025-10-1-5-16

[7] Rudakov IA. On the existence of a countable number of periodic solutions of a boundary value problem for the equation of beam vibrations with homogeneous boundary conditions. Differ Equ. 2022;58(8):1062–72. DOI: 10.31857/S0374064122080064

[8] Irgashev BYu. Solution of the problem with initial conditions of Cauchy type for a high order equation with a fractional Hilfer derivative. Differ Equ. 2022;58(9):1205–19. DOI: 10.31857/S0374064122090047

[9] Muminov FM, Karimov SYa. On mixed boundary value problems for a third-order composite equation. Oriental Renaissance Innov Educ Nat Soc Sci. 2024;4(2):623–9.

[10] Bitsadze AV. Equations of mathematical physics. Moscow: Mir Publishers; 1989. 381 P.

[11] Krasnov ML, Kiselev AI, Makarenko GI. Integral equations: Problems and examples with detailed solutions. Moscow: Mir Publishers; 1971. 224 P.

[12] Sopuev AA. Boundary value problems for fourth-order equations and mixed-type equations [Doctoral dissertation]. Bishkek; 1996.
[13] Polyanin D. Handbook of linear equations of mathematical physics. Moscow: Fizmatlit; 2001. 576 P.
[14] Khodjaniyazov AG. Boundary value problem for a fourth-order equation with a spectral parameter. In: International scientific conference on nonclassical equations of mathematical physics and their applications. Tashkent: National University of Uzbekistan; 2024. P. 251.
[15] Apakov YuP, Mamajonov SM. Solvability of a boundary value problem for a fourth equations of parabolic-hyperbolic type in a pentagonal domain. J Appl Ind Math. 2021;15(4):586–96. DOI: 10.1134/S1990478921040025
[16] Klyachin AA, Verevkin IYu. Construction of C^1-smooth piecewise-quadratic functions in solving boundary-value problems of 4^{th}-order equations on a triangular mesh. Math Phys Comput Model. 2023;26(2):5–14. DOI: 10.15688/mpcm.jvolsu.2023.2.1
[17] Karachik VV. Solution to the Dirichlet problem for the polyharmonic equations in the ball. Sib Adv Math. 2022;32(3):197–210. DOI: 10.33048/mattrudy.2021.24.204
[18] Urinov AK, Azizov MS. On the solvability of the initial boundary value problem for a high even order equation degenerating on the boundary of a domain. Sib J Ind Math. 2023;26(2):155–70. DOI: 10.33048/SIBJIM.2023.26.213
[19] Urinov AK, Usmonov DA. On one problem for a fourth-order mixed-type equation that degenerates inside and on the boundary of a domain. Bull Udmurt Univ Math Mech Comput Sci. 2023;33(2):312–28. DOI: 10.35634/vm230209
[20] Balkizov ZhA. Local boundary value problems for a model equation of the third order of hyperbolic type. News Kabardino-Balkar Sci Cent Russ Acad Sci. 2022;5(109):11–8. DOI: 10.35330/1991-6639-2022-5-109-11-18
[21] Goziev KS. Boundary value problem for fourth order equations of mixed-composite type. Int J Educ Soc Sci Humanit. 2023;11(5):619–25.
[22] Bekiev AB, Eshmuratov EE. Initial-boundary value problem for a degenerate fourth-order equation. Acad Res Educ Sci. 2021;2(10):745–50. DOI: 10.24412/2181-1385-2021-10-745-750
[23] Abdullaev OKh, Matchanova AA. On the solvability of boundary-value problems for third-order equations of parabolic-hyperbolic type with lower terms. Results Sci Technol Mod Math Its Appl Themat Rev. 2022;210:12–23. DOI: 10.36535/0233-6723-2022-210-12-23
[24] Mironov N. Construction of the Riemann Hadamard function for the three-dimensional Bianchi equation. Proc High Educ Inst Math. 2021;65(3):76–82. DOI: 10.26907/0021-3446-2021-3-76-82

Төртүнчү даражадагы курамдуу жана гиперболикалык типтеги теңдеме үчүн эки түр өзгөрүү сызыгы менен сопряжение маселеси

Сайфулло Бабаев
Физика-математика илимдеринин кандидаты, доцент
Тажикстан техникалык университетинин филиалы
735920, А. Джураев көч., 52, Исфара ш., Тажикстан Республикасы
https://orcid.org/0009-0008-6971-8009
Замирбек Бекмаматов
Физика-математика илимдеринин кандидаты, доцент
Баткен мамлекеттик университети
720100, И. Жусупов көч., 11, Баткен ш., Кыргыз Республикасы
https://orcid.org/0009-0003-3325-1519

Аннотация. Жогорку даражадагы аралаш жана аралаш-курамдуу типтеги теңдемелер үчүн чек шарттуу маселелер жылуулуктун, толкундардын таралышы жана аз илешкек чөйрөлөрдүн кыймылы менен байланышкан кубулуштарды математикалык моделдөөнүн маанилүү аспектисин түзөт. Изилдөөнүн актуалдуулугу теңдеменин түрү өзгөргөн сызыктар жана татаал чек шарттар болгон шарттарда мындай маселелерди так анализдөөнүн зарылчылыгы менен негизделет. Иштин максаты – теңдеменин түзүлүшү ар түрдүү болгон үч подобласка бөлүнгөн областта төртүнчү даражадагы курамдуу жана гиперболикалык типтеги теңдеме үчүн чек шарттуу маселени формулировкалоо жана ар тараптуу изилдөө болду. Маселе тиешелүү подобластардагы үч кошумча подмаселеге келтирилип, бул учурда теңдеменин түрү өзгөргөн сызыктарда экинчи даражага чейинки туундулар аркылуу берилген издөөгө тийиш болгон функция үчүн сопряжение шарттары киргизилди. Изилдөөдө чек шарттуу маселелер теориясынын классикалык ыкмалары, теңдеменин даражасын төмөндөтүү ыкмасы, ошондой эле аралаш-курамдуу типтеги теңдемелер теориясынын ыкмалары колдонулган. Ар бир кошумча маселе стандарттык формаларга – Дирихле, Гурса жана Дарбу маселелерине келтирилген. Түрү өзгөргөн сызыктарда экинчи даражадагы дифференциалдык теңдемелер алынган жана алар үчүн так курулган Грин функциялары аркылуу чек шарттуу маселелер чечилген. Гиперболикалык подмаселелер Вольтерра жана Фредгольм интегралдык теңдемелеринин экинчи түрүнө редукцияланып, алардын өзгөчө чечилүү шарты катары ядролорду баалоо негизинде жетиштүү шарттар алынган. Натыйжада ар бир подобласт үчүн чечимдин так аналитикалык формулалары алынды. Алынган жыйынтыктар гетерогендик чөйрөлөрдөгү процесстерди талдоодо жана математикалык физика маселелеринде сандык моделдерди курууда колдонулушу мүмкүн

Негизги сөздөр: чек шарттуу маселелер; Дирихле маселеси; Дарбу типтеги маселе; Грин функциясы; сопряжение шарттары; чек шарттар; Вольтерра жана Фредгольм тендемелери

Задача сопряжения для уравнения составного и гиперболического типов четвертого порядка с двумя линиями изменения типа

Сайфулло Бабаев
Кандидат физико-математических наук, доцент
Филиал технологического университета Таджикистана
735920, ул. А. Джураева, 52, г. Исфара, Республика Таджикистан
https://orcid.org/0009-0008-6971-8009

Замирбек Бекмаматов
Кандидат физико-математических наук, доцент
Баткенский государственный университет
720100, ул. И. Жусупова, 11, г. Баткен, Кыргызская Республика
https://orcid.org/0009-0003-3325-1519

Аннотация. Краевые задачи для уравнений смешанного и смешанно-составного типов высокого порядка играют важную роль в математическом моделировании явлений, связанных с распространением тепла, волн и движением слабовязких сред. Актуальность исследования обусловлена необходимостью строгого анализа таких задач, особенно в условиях наличия линий смены типа уравнения и сложных граничных условий. Целью работы было формулирование и всестороннее исследование краевой задачи для уравнения четвёртого порядка составного и гиперболического типов в области, разделённой на три подобласти с различной структурой уравнений. Задача была приведена к трём вспомогательным подзадачам в соответствующих подобластях, при этом на линиях изменения типа уравнений вводились условия сопряжения, выраженные через искомую функцию и её производные до второго порядка. Использовались классические методы теории краевых задач, приём понижения порядка уравнений, а также методы теории уравнений смешанно-составного типа. Каждая вспомогательная задача была сведена к стандартным постановкам – задачам Дирихле, Гурса и Дарбу. На линиях смены типа получены дифференциальные уравнения второго порядка, для которых решены краевые задачи с использованием явно построенных функций Грина. Гиперболические подзадачи редуцированы к интегральным уравнениям Вольтерра и Фредгольма второго рода, и получены достаточные условия их однозначной разрешимости через оценки ядер. В результате получены явные аналитические выражения решений в каждой подобласти. Результаты могут быть применены для анализа процессов в неоднородных средах и при построении численных моделей в задачах математической физики

Ключевые слова: краевые задачи; задача Дирихле; задача типа Дарбу; функция Грина; условия согласования; краевые условия; уравнения Вольтерра и Фредгольма

VOL. 4, No. 1
Journal of Osh State University. Mathematics. Physics. Technical Sciences
2025
Journal of Osh State University
Journal homepage: https://mpht-oshsu.com/en

UDC 517.956.6
DOI: 10.52754/16948645_2025_4(1)_30

Article's History:
Received 14.02.2025 Revised 16.05.2025 Accepted 19.06.2025

Boundary value problems for a mixed type equation of the 3rd order with variable coefficients

Adakhimzhan Sopuev*
Doctor of Physical and Mathematical Sciences, Professor
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0009-0008-8530-6629

Baktybek Nuranov
Senior Lecturer
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0009-0003-4120-8780

Abstract. The existence and uniqueness of the solution to the boundary value problem for a third-order mixed-type equation with variable coefficients in the lower terms is proven, with conditions for the gluing of the function itself and its first- and second-order derivatives on the line $y = 0$, where the type of equation changes when a second-order mixed parabolic-hyperbolic operator is applied to a first-order linear differential operator with constant coefficients. By reducing the order of the equation, the problem was reduced to the Tricomi problem for a second-order mixed parabolic-hyperbolic equation with continuous conditions for the function itself and its first-order derivative with respect to y on the line of change of the equation type. By the method of elimination of the system of equations obtained from the parabolic and hyperbolic parts of the domains, the solvability of the problem was reduced to the solvability of the Fredholm integral equation of the second kind. A sufficient condition for the solvability of the integral equation was obtained through estimates of the kernel of the equation. The solution of the problem was split into two problems in the regions under consideration: in the parabolic part of the region, the first boundary value problem for the heat conduction equation was solved using the Green's function method, and in the hyperbolic part of the region, bounded by the characteristics of the equation and the line $y = 0$, the solution of the problem using the Riemann function construction method was determined as the solution of the Cauchy problem. By applying a curvilinear integral, the solution to the problem in the areas under consideration was found. The necessity of the requirement of continuity of the function itself and its first two derivatives with respect to y on the line of change of the equation type was justified. Sufficient conditions for the unique classical solvability of the boundary value problem were established. The obtained conditions for the solvability of the boundary value problem provided a theoretical basis for the development of numerical methods for solving applied problems in aerohydrodynamics, geophysics, and engineering thermodynamics

Keywords: existence; uniqueness; Green's function; Riemann's function; integral equation; order reduction method

Introduction

Mathematical modeling of a number of applied problems in fluid mechanics, physics, and mathematical biology has led to the formulation and investigation of the solvability of boundary value problems with nonlocal and integral conditions for third-order mixed-type equations, since in practice, the measurement of certain characteristics of the desired function and its derivatives is possible only in averaged or integral form.

The formulation of well-posed boundary value problems for partial differential equations and their

Suggested Citation:
Sopuev A, Nuranov B. Boundary value problems for a mixed type equation of the 3rd order with variable coefficients. J Osh State Univ Math Phys Tech Sci. 2025;4(1):30–40. DOI: 10.52754/16948645_2025_4(1)_30

*Corresponding author

investigation are urgent tasks of the modern theory of partial differential equations. Despite the wide application of mixed-type equations in problems of gas dynamics, hydrodynamics, and other applied disciplines, boundary value problems for third-order mixed-type equations with two independent variables remain insufficiently studied, which necessitates further research in this direction.

K. Belakroum [1] studied a nonlocal boundary value problem for third-order partial differential equations in Hilbert space with a self-adjoint positively defined operator, in which stability estimates for the solution of two nonlocal problems for third-order partial differential equations were obtained.

In the works of R.R. Ashurov & Yu.E. Fayziev [2], the uniqueness and existence of the solution of the inverse problem of determining the order of the fractional derivative with respect to time in an inhomogeneous subdiffusion equation with an arbitrary elliptic differential operator with constant coefficients in an n-dimensional torus was proved. Using the classical Fourier method, it was proved that the value of the solution at a fixed moment in time, based on observational data, uniquely reconstructs the order of the fractional derivative.

Zh.A. Balkizov [3] investigated a boundary value problem with shift for a third-order inhomogeneous parabolic-hyperbolic equation with a wave operator in the hyperbolic region, when a linear combination of the values of the desired function on two independent characteristics and on the line of type change is given as a boundary condition. Necessary and sufficient conditions for the existence and uniqueness of a regular solution to the problem were found.

N.K. Ochilova & T.K. Yuldashev [4] investigated the existence and uniqueness of the solution of a nonlocal boundary value problem for a degenerate differential equation of mixed type. A parabolic-hyperbolic equation with a Gerasimov-Caputo fractional derivative was considered. The uniqueness of the solution was proved by the method of energy integrals using some properties of hypergeometric functions and integro-differential operators of fractional order. The existence of a solution is proved by the method of integral equations.

The solvability in anisotropic Sobolev spaces of nonlocal boundary value problems for third-order pseudoparabolic equations was studied by A.I. Kozhanov & G.I. Tarasova [5]. A feature of the problems studied is that they impose a condition on the spatial variable that combines the generalised Samarskii-Ionkin condition and an integral type condition. The purpose of the work was to prove the existence and uniqueness of regular solutions to the problems studied – solutions having all generalised Sobolev derivatives included in the corresponding equation.

In the work of D.K. Durdieva & Sh.B. Merajova [6], an inverse problem related to finding an unknown right-hand side was studied for a mixed parabolic-hyperbolic equation with a Bessel operator. Based on the method of separation of variables, the problem is reduced to solving ordinary differential equations with respect to the coefficients of the expansion into Bessel-Fourier series of unknown functions in terms of orthonormal Bessel functions of the first kind of zero order. A criterion for the uniqueness and existence of a solution to the posed problem was established.

In the work of S.N. Sidorov [7], an initial-boundary value problem was investigated for an inhomogeneous mixed parabolic-hyperbolic equation of three variables with a degenerating parabolic part in a rectangular parallelepiped. A criterion for the uniqueness of the solution was established. The solution was constructed as a sum of an orthogonal series. The stability of the solution with respect to the boundary function and the right-hand side of the equation was established.

In the work of R.Kh. Makaova [8], a theorem on the existence and uniqueness of a regular solution to a mixed boundary value problem for a third-order hyperbolic equation with order degeneration inside the domain was proved. In the positive part of the domain, the considered equation coincided with the Aller equation, which is a third-order hyperbolic equation. And in the negative part of the domain, it coincided with a degenerate hyperbolic equation of the first kind. The uniqueness of the solution of the studied problem was proved by the Tricomi method.

In the work of D.K. Durdiev [9], direct and inverse problems for a model mixed parabolic-hyperbolic equation were considered. In the direct problem, an analog of the Tricomi problem for this equation with a characteristic line of type change was considered. The unknown in the inverse problem is the variable coefficient at the lower term of the parabolic equation. For its determination with respect to the solution defined in the parabolic part of the domain, an integral overdetermination condition is given. Local theorems on the unique solvability of the posed problems in the sense of a classical solution are proved.

The purpose of this work was to formulate and investigate new well-posed boundary value problems for third-order equations, when a mixed parabolic-hyperbolic operator is applied to a first-order differential operator of a certain form. The questions of where and what mandatory boundary conditions should be set for a well-posed problem, the number of gluing conditions on the line of change of the equation type, and what methods should be used to solve the problem were unknown beforehand, so they were determined during the study of the problem.

Materials and Methods

Let C^{n+m} means the class of functions having all continuous derivatives $\partial^{r+s}/\partial x^r \partial y^s (r = 0,1,..., n;\ s = 0,1,..., m)$. In this work, a boundary value problem is considered where a mixed parabolic-hyperbolic operator is applied to a linear first-order differential operator.

Problem Statement. In the domain D, limited by line segments AC: $x+y=0$, CB: $x-y=l$, BB_0: $x=l$, B_0A_0: $y=h$, B_0A: $x=0(l,h>0)$, the equation is considered:

$$L_1L_2u=0, \quad (1)$$

$$L_1 \equiv \begin{cases} \frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial y} + c_1(x,y), y>0, \\ \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + a_2(x,y)\frac{\partial}{\partial x} + b_2(x,y)\frac{\partial}{\partial y} + c_2(x,y), y<0, \end{cases}$$

$$L_2 \equiv \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y},$$

where $a_2(x,y)$, $b_2(x,y)$, $c_1(x,y)$, $c_2(x,y)$ – given functions satisfying the following smoothness conditions:

$$c_1(x,y) \in C(\overline{D}_1), \forall (x,y) \in \overline{D}_1: c_1(x,y) \le 0,$$

$$a_2(x,y),\ a_{2x}(x,y),\ a_{2y}(x,y),$$
$$b_2(x,y),\ b_{2x}(x,y),\ b_{2y}(x,y), c_2(x,y) \in C(\overline{D}_2). \quad (2)$$

Let $D_1=D\cap(y>0)$, $D_2=D\cap(y<0)$. The characteristic equation for equation (1) in the domain D_1 has the form $-(dy)^3+(dy)^2dx=0$. Consequently, the line $y=const$ is a 2-fold characteristic, $x-y=const$ a single characteristic of equation (1) in the domain D_1. The characteristic equation for equation (1) in the domain D_2 has the form $-(dy)^3+(dy)^2dx+dy(dx)^2-(dx)^3=0$, which can be written as $(dx+dy)(dx-dy)^2=0$. Consequently, the line $x-y=const$ is a 2-fold characteristic, and $x+y=const$ a single characteristic of equation (1) in the domain D_2. Thus, equation (1) in the domain D_1 belongs to the hyperbolic type, and in the domain D_2 – also hyperbolic type. This fact means that equation (1) is a mixed-type equation in the domain D, since when crossing the lines $y=0$ the characteristics of equation (1) differ, which affects the well-posedness of the problem. This paper considers the formulation and investigation of the well-posedness of the following problem.

Problem 1. It is required to determine the function $u(x,y)$ with the following properties:

1) $u(x,y) \in C(\overline{D}) \cap C^2(D) \cap [C^{3+2}(D_1) \cup C^{3+3}(D_2)]$;
2) $u(x,y)$ is a solution of equation (1) in the domain $D\setminus(y=0)$;
3) $u(x,y)$ satisfies the following boundary conditions:

$$u(0,y)=\varphi_1(y),\ u(l,y)=\varphi_2(y),\ 0\le y\le h, \quad (3)$$

$$u_x(0,y)=\varphi_3(y),\ u_x(l,y)=\varphi_4(y),\ 0\le y\le h, \quad (4)$$

$$\left.\frac{\partial u}{\partial n}\right|_{AC} = \psi(x), 0 \le x \le \frac{l}{2}, \quad (5)$$

where $\varphi_i(y)(i=\overline{1,4})$, $\psi(x)$ – given smooth functions, n – inner normal, and:

$$\varphi_i(y) \in C^2[0,h](i=1,2), \varphi_j(y) \in C^1[0,h]$$
$$(j=3,4), \psi(x) \in C^2\left[0,\frac{l}{2}\right], \quad (6)$$

$$\varphi_3(0)-\varphi_1'(0)=-\sqrt{2}\psi(0). \quad (7)$$

From the statement of Problem 1, the following gluing conditions follow:

$$u(x,-0)=u(x,+0)=\tau(x), u_y(x,-0)=$$
$$=u_y(x,+0)=\nu(x), 0\le x\le l,$$
$$u_{yy}(x,-0)=u_{yy}(x,+0)=\mu(x), 0\le x\le l, \quad (8)$$

where $\tau(x)$, $\nu(x)$, $\mu(x)$ – currently unknown functions.

By the method of reducing the order of equations, Problem 1 is reduced to an analogue of the Tricomi problem for a mixed parabolic-hyperbolic equation with continuous gluing conditions with the line of change of type $y=0$ in the following way. The solution to Problem 1 was considered separately in each of the domains. D_1 and D_2. Equation (1) in the domain D_1 is written as a system of equations:

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \upsilon_1(x,y), (x,y) \in D_1, \\ \frac{\partial^2 \upsilon_1}{\partial x^2} - \frac{\partial \upsilon_1}{\partial y} + c_1(x,y)\upsilon_1 = 0, (x,y) \in D_1, \end{cases} \quad (9)$$

and in the domain D_2 – in the form of:

$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \upsilon_2(x,y), (x,y) \in D_2, \\ \frac{\partial^2 \upsilon_2}{\partial x^2} - \frac{\partial^2 \upsilon_2}{\partial y^2} + a_2(x,y)\frac{\partial \upsilon_2}{\partial x} + b_2(x,y)\frac{\partial \upsilon_2}{\partial y} + \\ \quad + c_2(x,y)\upsilon_2 = 0, (x,y) \in D_2. \end{cases} \quad (10)$$

From the gluing condition (8) on the line $y=0$ the equalities follow: $\upsilon_1(x,+0)=\tau'(x)-\nu(x)$, $\upsilon_2(x,-0)=\tau'(x)-\nu(x)$. Therefore, according to the first two conditions (8), the equality holds: $\upsilon_1(x,-0)=\upsilon_2(x,+0)$. By differentiating the first equation of system (9) with respect to y the equality is obtained: $\frac{\partial^2 u}{\partial x\partial y} - \frac{\partial^2 u}{\partial y^2} = \frac{\partial \upsilon_1(x,y)}{\partial y}$. Then at $y=0$ taking into account the gluing condition (8) the equality follows: $\frac{\partial^2 u(x,+0)}{\partial x\partial y} - \frac{\partial^2 u(x,+0)}{\partial y^2} = \frac{\partial \upsilon_1(x,+0)}{\partial y}$, which can be written as: $\frac{\partial \upsilon_1(x,+0)}{\partial y} = \nu'(x) - \mu(x)$. Similarly, from the first equation of system (10), the equality is obtained: $\frac{\partial \upsilon_2(x,-0)}{\partial y} = \nu'(x) - \mu(x)$. Consequently, according to the second and third conditions of (8), the equality holds: $\frac{\partial \upsilon_1(x,+0)}{\partial y} = \frac{\partial \upsilon_2(x,-0)}{\partial y}$. Thus, for functions $\upsilon_1(x,y)$ and $\upsilon_2(x,y)$ the following gluing conditions hold on the line $y=0$:

$$\upsilon_1(x,+0)=\upsilon_2(x,-0)=\nu_1(x), \frac{\partial \upsilon_1(x,+0)}{\partial y} = \frac{\partial \upsilon_2(x,-0)}{\partial y} = \mu_1(x), \quad (11)$$

where $\nu_1(x)$ and $\mu_1(x)$ – currently unknown functions. The functions $\nu_1(x)$ and $\mu_1(x)$ are related to the functions $\tau(x)$, $\nu(x)$ and $\mu(x)$ as follows:

$$\nu_1(x)=\tau'(x)-\nu(x), \mu_1(x)=\nu'(x)-\mu(x). \quad (12)$$

Thus, the equations

$$\frac{\partial^2 \upsilon_1}{\partial x^2} - \frac{\partial \upsilon_1}{\partial y} + c_1(x,y)\upsilon_1 = 0, (x,y) \in D_1, \quad (13)$$

$$\frac{\partial^2 \upsilon_2}{\partial x^2}-\frac{\partial^2 \upsilon_2}{\partial y^2}+a_2(x,y)\frac{\partial \upsilon_2}{\partial x}+b_2(x,y)\frac{\partial \upsilon_2}{\partial y}+$$
$$+c_2(x,y)\upsilon_2=0,(x,y)\in D_2, \quad (14)$$

are related by the conjugation conditions (11). Therefore, to determine the functions $\upsilon_1(x,y)$ and $\upsilon_2(x,y)$ the following auxiliary problem was considered.

Problem 2. It is required to determine the function $\upsilon_1(x,y)\in C(\overline{D}_1)\cap C^1(\overline{D}_1)\cap C^{2+1}(D_1)$ and $\upsilon_2(x,y)\in C(\overline{D}_2)\cap$ $\cap C^1(\overline{D}_2)\cap C^2(D_2)$, satisfying the following conditions:

1) $\upsilon_1(x,y)$ is a solution of equation (13) in the domain. D_1, $\upsilon_2(x,y)$ is a solution of equation (14) in the domain D_2;

2) $\upsilon_1(x,y)$ satisfies the boundary conditions:

$$\upsilon_1(0,y)=\tilde{\varphi}_1(y),\ \upsilon_2(l,y)=\tilde{\varphi}_2(y),\ 0\le y\le l, \quad (15)$$

3) $\upsilon_2(x,y)$ satisfies the boundary condition

$$\upsilon_2(x,-x)=\tilde{\psi}(x),\ 0\le x\le \frac{l}{2}, \quad (16)$$

4) $\upsilon_1(x,y)$ *and* $u\upsilon_2(x,y)$ satisfy the gluing conditions (11), where $\tilde{\varphi}_1(y)=\varphi_3(y)-\varphi_1'(y)$, $\tilde{\varphi}_2(y)=\varphi_4(y)-\varphi_2'(y)$, $\tilde{\psi}(x)=-\sqrt{2}\psi(x)$.

To solve Problem 2, it is first necessary to find the functions $\nu_1(x)$ and $\mu_1(x)$. To determine these functions, relationships obtained from both domains are required D_1, as well as from the domain D_2. To obtain a relationship from the domain D_1 between $\nu_1(x)$ and $\mu_1(x)$ a limiting transition from the equation is used (13) at $y\to+0$, and to obtain the second relationship between $\nu_1(x)$ and $\mu_1(x)$. The general representation of the solution to the Cauchy problem for equation (14), presented through Riemann functions, is used [10].

Results and Discussion

First, Problem 2 was considered. To solve Problem 2, it is necessary to obtain relationships derived from both regions D_1, and also from the domain D_2.

Obtaining a relationship from the domain D_1. When tending towards y to +0 from equation (13), a relationship between the functions follows. $\nu_1(x)$ and $\mu_1(x)$, from equation (13), a relationship between the functions follows. D_1:

$$\nu_1''(x)+c_1(x,0)\nu_1(x)-\mu_1(x)=0,0\le x\le l. \quad (17)$$

From the boundary conditions (15), the following conditions are obtained:

$$\nu_1(0)=\tilde{\varphi}_1(0),\ \nu_1(l)=\tilde{\varphi}_2(0). \quad (18)$$

If introduce the notation:

$$\nu_1(x)=\tilde{\varphi}_1(0)+\frac{x}{l}[\tilde{\varphi}_2(0)-\tilde{\varphi}_1(0)]+\nu_2(x), \quad (19)$$

where $\nu_2(x)$ – a new unknown function, then from (17) the relation is obtained:

$$\nu_2''(x)+c_1(x,0)\ \nu_2(x)=\mu_1(x)+\Phi_1(x),\ 0\le x\le l, \quad (20)$$

where $\Phi_1(x)=-c_1(x,\ 0)\{\tilde{\varphi}_1(0)+\frac{x}{l}\ [\tilde{\varphi}_2(0)-\tilde{\varphi}_1(0)]\}$. The boundary conditions are then of the form:

$$\nu_2(0)=0,\ \nu_2(l)=0. \quad (21)$$

The following lemma holds.

Lemma 1. If the conditions are met: $\forall x\in[0,\ l]$: $c_1(x,\ 0)\le 0$, (22) then the homogeneous problem (20)-(21) has only a trivial solution.

Proof. After multiplying the homogeneous equation $\nu_2''(x)+c_1(x,\ 0)\ \nu_2(x)=0$ at $\nu_2(x)$ and of integrating the obtained relation from 0 to l taking into account the homogeneous conditions (21) the equality holds:

$$\int_0^l \nu_2(x)[\nu_2''(x)+c_1(x,\ 0)\ \nu_2(x)]dx=$$
$$=\int_0^l\{-[\nu_2'(x)]^2+c_1(x,\ 0)[\nu_2(x)]^2\}dx=0.$$

It is obvious that if the condition is met (22), then $\forall x\in[0,l]$: $\nu_2(x)\equiv 0$. Lemma 1 is proved. Solution of the inhomogeneous equation (20), satisfying the homogeneous boundary conditions (21), has the form:

$$\nu_2(x)=\Phi_2(x)+\int_0^l G(x,\ \xi)\ \mu_1(\xi)d\xi,\ 0\le x\le l, \quad (22)$$

where $\Phi_2(x)=\int_0^l G(x,\ \xi)\ \Phi_1(x)dx$, $G(x,\ \xi)$ – Green's function. Then equality (19) will be written as:

$$\nu_1(x)=\Phi(x)+\int_0^l G(x,\ \xi)\ \mu_1(\xi)d\xi,\ 0\le x\le l, \quad (23)$$

where $\Phi(x)=\tilde{\varphi}_1(0)+\frac{x}{l}\ [\tilde{\varphi}_2(0)-\tilde{\varphi}_1(0)]+\Phi_2(x)$. Equality (23) represents the relationship between functions $\nu_1(x)$ and $\mu_1(x)$ obtained from the domain D_1.

Obtaining a relationship from the domain D_2. For the hyperbolic equation (14) in the domain D_2. The Cauchy problem with conditions is considered:

$$\upsilon_2(x,0)=\nu_1(x),\ \upsilon_{2y}(x,0)=\mu_1(x),\ 0\le x\le l, \quad (24)$$

the solution of which is presented in the form:

$$\upsilon_2(x,y)=\frac{1}{2}[R(x,y;\ x+y,\ 0)\ \nu_1(x+y)+R(x,y;\ x-y,\ 0)\times$$
$$\times\nu_1(x-y)]+\frac{1}{2}\int_{x+y}^{x-y}R_\eta(x,y;\ \xi,\ 0)+b_2(\xi,\ 0)\ R(x,y;\ \xi,\ 0)]\nu_1(\xi)\times$$
$$\times d\xi-\frac{1}{2}\int_{x+y}^{x-y}R(x,y;\ \xi,\ 0)\ \mu_1(\xi)d\xi, \quad (25)$$

where $R(x,\ y;\ \xi,\ \eta)$ – Riemann function [11]. This function is defined as the solution to the Goursat problem for the equation:

$$R_{\xi\xi}-R_{\eta\eta}-(a_2R)_\xi-(b_2R)_\eta+c_2R=0, \quad (26)$$

satisfying the conditions:

$$R(x,y;\xi,x+y-\xi)=$$
$$=exp\left\{-\frac{1}{2}\int_\xi^x[a_2(t,x+y-t)+b_2(t,x+y-t)]\,dt\right\}, \quad (27)$$

$$R(x,y;\xi,\xi-x+y)=$$
$$=exp\left\{\frac{1}{2}\int_x^{\xi}[a_2(t,t-x+y)-b_2(t,t-x+y)]\,dt\right\},(28)$$

$$R(x,y;x,y)=1. \qquad (29)$$

After using condition (16), equality (28) is written as:

$$R(x,-x;2x,0)\nu_1(2x)=2\tilde{\psi}(x)-R(x,-x;0,0)\nu_1(0)-$$
$$-\int_0^{2x}[R_\eta(x,-x;\xi,0)+b_2(\xi,0)R(x,-x;\xi,0)]\,\nu_1(\xi)d\xi+$$
$$+\int_0^{2x}R(x,-x;\xi,0)\,\mu_1(\xi)d\xi,\ 0\le x\le\frac{l}{2}. \qquad (30)$$

Obviously, that $\forall x\in[0,\frac{l}{2}]$: $0\le 2x\le l$. Taking into account that $\nu_1(0)=\tilde{\psi}(0)$, then from (30), upon substitution $2x=z$ and then z at x the relation is obtained:

$$R\left(\frac{x}{2},-\frac{x}{2};x,0\right)\nu_1(x)=$$
$$=-\int_0^x\left[R_\eta\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)+b_2(\xi,0)R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)\right]\times$$
$$\times\nu_1(\xi)d\xi+\int_0^x R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)\mu_1(\xi)d\xi+$$
$$+2\tilde{\psi}\left(\frac{x}{2}\right)-R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)\tilde{\psi}(0). \qquad (31)$$

Lemma 2. $\forall x\in[0,l]$ the inequality holds:

$$R\left(\frac{x}{2},-\frac{x}{2};x,0\right)>0. \qquad (32)$$

Proof. Let $\xi = x - y$. Then equality (28) will take the form $R(x,y;x-y,0)=exp\left\{\frac{1}{2}\int_x^{x-y}\begin{bmatrix}a_2(t,t-x+y)-\\-b_2(t,t-x+y)\end{bmatrix}dt\right\}$. If $y = -x$, then this equality will be written as $R(x,-x;2x,0)=exp\left\{\frac{1}{2}\int_x^{2x}[a_2(t,t-2x)--b_2(t,t-2x)]\,dt\right\}$. When replacing $2x$ to z, then z to x, the following relation holds $R\left(\frac{x}{2},-\frac{x}{2};x,0\right)=exp\left\{\frac{1}{2}\int_{\frac{x}{2}}^{x}[a_2(t,t-x)-b_2(t,t-x)]\,dt\right\}$. From this equality, the validity of inequality (32) follows. Lemma 2 is proved.

If take into account inequality (32) from Lemma 2, then it follows from (31) that the relation between functions $\nu_2(x)$ and $\mu_2(x)$, obtained from the domain D_2, is presented as:

$$\nu_1(x)=\int_0^x P_1(x,\xi)\,\nu_1(\xi)d\xi+$$
$$+\int_0^x P_2(x,\xi)\,\mu_1(\xi)d\xi+\psi_1(x), \qquad (33)$$

where $\quad P_1(x,\xi)=-\frac{\left[R_\eta\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)+b_2(\xi,0)R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)\right]}{R\left(\frac{x}{2},-\frac{x}{2};x,0\right)}.$

$$\frac{P_2(x,y)=R\left(\frac{x}{2},-\frac{x}{2};\xi,0\right)}{R\left(\frac{x}{2},-\frac{x}{2};x,0\right)},\quad \psi_1(x)=\frac{\left[2\tilde{\psi}\left(\frac{x}{2}\right)-R\left(\frac{x}{2},-\frac{x}{2};0,0\right)\tilde{\psi}(0)\right]}{R\left(\frac{x}{2},-\frac{x}{2};x,0\right)}.$$

After inverting the Volterra part of equation (33), the relation is obtained:

$$\nu_1(x)=\int_0^x K_1(x,\xi)\,\mu_1(\xi)d\xi+\Psi_1(x), \qquad (34)$$

where $K_1(x,\xi)=P_2(x,\xi)+\int_\xi^x G(x,t)\,P_2(t,\xi)dt$, $\Psi_1(x)=\psi_1(x)+\int_0^x G(x,t)\,\psi_1(t)dt$, $G(x,\xi)$ – resolvent of the kernel $P_1(x,\xi)$. Equality (34) represents the relationship between the functions $\nu_1(x)$ and $\mu_1(x)$, obtained from the domain D_2.

Reduction of the problem to an integral equation. After eliminating $\nu_1(x)$ From equations (23) and (34), the following relationship is obtained:

$$\int_0^x K_1(x,\xi)\,\mu_1(\xi)d\xi=\int_0^l G(x,\xi)\mu_1(\xi)d\xi+\Psi_2(x)\ ,(35)$$

where $\Psi_2(x)=\Phi(x)-\Psi_1(x)$. After differentiating equation (35), the following equation is obtained:

$$K_1(x,x)\mu_1(x)+\int_0^x K_{1x}(x,\xi)\,\mu_1(\xi)d\xi=$$
$$=\int_0^l G_x(x,\xi)\mu_1(\xi)d\xi+\Psi_2'(x). \qquad (36)$$

This is $\forall x\in[0,l]$: $K_1(x,x)=P_2(x,x)=1$, then equation (36) is written as:

$$\mu_1(x)=\int_0^x[-K_{1x}(x,\xi)]\mu_1(\xi)d\xi+$$
$$+\int_0^l G_x(x,\xi)\mu_1(\xi)d\xi+\Psi_2'(x). \qquad (37)$$

After inverting the Volterra part of equation (37), a Fredholm integral equation of the second kind is obtained:

$$\mu_1(x)=\Psi(x)+\int_0^l K(x,\xi)\,\mu_1(\xi)d\xi, \qquad (38)$$

where $K(x,\xi)=G_x(x,\xi)+\int_0^x R_2(x,t)\,G_x(t,\xi)dt$, $\Psi(x)==\Psi_2'(x)+\int_0^x R_2(x,t)\,\Psi_2'(t)dt$, and $R_2(x,t)$ – resolvent of the kernel $-K_{1x}(x,\xi)$. Let $\|K\|_{C(\bar{Q})}=\max\limits_{(x,\xi)\in\bar{Q}}|K(x,\xi)|$, where $Q=\{(x,\xi): 0\le x\le l,\ 0\le\xi\le l\}$. If:

$$l\cdot\|K\|_{C(\bar{Q})}<1, \qquad (39)$$

then the integral equation (38) has a unique solution [12]. The solution of equation (38) can be represented as:

$$\mu_1(x)=\Psi(x)+\int_0^l R(x,\xi)\,\Psi(\xi)d\xi, \qquad (40)$$

where $R(x,\xi)$ – resolvent of the kernel $K(x,\xi)$. Then from (23) is also determined $\nu_1(x)$:

$$\nu_1(x)=\Phi(x)+\int_0^l G(x,\xi)\Psi(\xi)d\xi+$$
$$+\int_0^l G(x,\xi_1)d\xi_1\int_0^l R(\xi_1,\xi)\Psi(\xi)d\xi. \qquad (41)$$

Then the function $\upsilon_2(x,y)$, presented by formula (25), is fully determined, since the functions $\nu_1(x)$ and $\mu_1(x)$ are determined by formulas (40) and (41), respectively.

Solution of the problem in the domain D_1. In the domain D_1. The following problem is considered.

Problem 3. Find in the domain D_1 the solution of equation (13), satisfying the conditions:

$$\upsilon_1(0,y)=\tilde{\varphi}_1(y),\ \upsilon_1(l,y)=\tilde{\varphi}_2(y),$$
$$0\le y\le l;\ \upsilon_1(x,0)=\nu_1(x),0\le x\le l,$$

where $\nu_1(x)$ determined by the formula (41), and $\nu_1(0)=\tilde{\varphi}_1(0)$, $\nu_1(l)=\tilde{\varphi}_2(0)$

By the Green's function method, the solution of Problem 3 is reduced to solving a Volterra integral equation of the second kind:

$$\upsilon_1(x,y) = \upsilon_0(x,y) + \int_0^y d\eta \int_0^l N(x,y;\xi,\eta)\,\upsilon_1(\xi,\eta)d\xi, \quad (42)$$

where $(x, y; \xi, \eta) = -c_1(\xi, \eta)\ G(x, y; \xi, \eta)$, $G(x, y; \xi, \eta) = \frac{1}{2\sqrt{\pi(y-\eta)}}\sum_{n=-\infty}^{+\infty}\left\{exp\left[-\frac{(x-\xi+2nl)^2}{4(y-\eta)}\right] - exp\left[-\frac{(x+\xi+2nl)^2}{4(y-\eta)}\right]\right.$ – Green's function [13], and $\upsilon_0(x, y) = \int_0^y G\xi(x, y; 0, \eta)\ \tilde{\varphi}_1(\eta)d\eta - \int_0^y G\xi(x, y; l, \eta)\ \tilde{\varphi}_2(\eta)d\eta + \int_0^l G(x, y; \xi, 0)\ \nu_1(\xi)d\xi$ – known function. Since the kernel $N(x,y;\xi,\eta)$ has a weak singularity, therefore equation (42) has a unique solution, consequently Problem 3 is uniquely solvable in the domain D_1.

Problem 4. Find the solution to the equation in the domain

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \upsilon_1(x,y),\ (x,y) \in D_1, \quad (43)$$

satisfying conditions (3).

For solving Problem 4, a curvilinear integral of the second kind was used. On the plane $\xi O \eta$ the domain D_1 is divided into two regions: $D_1 = D_{11} \cup D_{12}$, where $D_{11} = \{(\xi,\eta): 0<\xi<l, \xi<\eta<l\}$, $D_{12} = \{(\xi,\eta): 0<\xi<l, 0<\eta<\xi\}$. In the domain D_{11} an arbitrary point is chosen $M_1(x,y)$ and a straight line is drawn $\eta = \xi - x + y$ through this point, which is parallel to the bisector $\eta = \xi$ of the first coordinate angle. This line intersects the y-axis at point $N_1(0, y-x)$. Further, on the segment N_1M_1 a curvilinear integral of the second kind is applied to equation (43):

$$\int_0^x [u_\xi(\xi,\xi-x+y) + u_\eta(\xi,\xi-x+y)]\,d\xi = \\ = \int_0^x \upsilon_1(\xi,\xi-x+y)d\xi.$$

This is $u_\xi(\xi, \xi-x+y) + u_\eta(\xi, \xi-x+y) = \frac{\partial}{\partial\xi}\, u(\xi, \xi-x+y)$, then after integrating the left side of the equality, taking into account the first boundary condition (3), the following relation holds:

$$u(x,y) = \varphi_1(y-x) + \int_0^x \upsilon_1(\xi, \xi-x+y)d\xi,\ (x,y) \in D_{11}.$$

Similarly, in the domain $D_{12} = \{(\xi,\eta): 0<\xi<l, 0<\eta<\xi\}$ an arbitrary point is considered $M_2(x, y)$ and a line $\eta = \xi - x + y$, parallel to the bisector $\eta = \xi$, which intersects the line $\xi = l$ at the point $N_2(l, l-x+y)$. Next, a curvilinear integral of the 2nd kind from the equation is calculated (43) on the segment M_2N_2:

$$\int_x^l [u_\xi(\xi,\xi-x+y) + u_\eta(\xi,\xi-x+y)]\,d\xi = \\ = \int_x^l \upsilon_1(\xi,\xi-x+y)d\xi.$$

Hence, as in the previous case, the solution to Problem 1, satisfying the second boundary condition (3), is determined as follows:

$$u(x,y) = \varphi_2(l-x+y) - \int_x^l \upsilon_1(\xi, \xi-x+y)d\xi,\ (x,y) \in D_{12}.$$

Thus, the solution to Problem 2 is presented as:

$$u(x,y) = \\ = \begin{cases} \varphi_1(y-x) + \int_0^x \upsilon_1(\xi,\xi-x+y)\,d\xi, (x,y) \in D_{11}, \\ \varphi_2(l-x+y) - \int_x^l \upsilon_1(\xi,\xi-x+y)\,d\xi, (x,y) \in D_{12}. \end{cases} \quad (44)$$

It should be noted that the solution to problem 4 in the field of D_1 may have a discontinuity of the first kind on the line $y = x$. If compliance with the agreement condition is required

$$\varphi_2(l) = \varphi_1(0) + \int_0^l \upsilon_1(\xi,\xi)d\xi, \quad (45)$$

then $u(x,y) \in C(\bar{D}_1)$ from (44). The trace of the function is defined on the segment AB:

$$\tau(x) = \varphi_2(l-x) - \int_x^l \upsilon_1(\xi, \xi-x)d\xi,\ 0 \le x \le l. \quad (46)$$

Solving problems in the field of D_2. In the domain D_2 considering the following problem.

Problem 5. Find in the domain D_2 equation solution

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \upsilon_2(x,y),\ (x,y) \in D_2, \quad (47)$$

satisfying the condition $u(x, 0) = \tau(x)$, $0 \le x \le l$.

Let $M_3(x, y)$. An arbitrary point of the region D_2. Through the point $M_3(x, y)$ the direct line is being built $\eta = \xi - x + y$, which is parallel to the line CB: $x - y = l$. This line intersects the x-axis at the point $N_3(x-y, 0)$. On the line $M_3N_3 = \{(x,y): x<\xi<x-y, \eta = \xi-x+y\}$. The line integral of the second kind is applied to equation (47):

$$\int_x^{x-y}[u_\xi(\xi, \xi-x+y) + u_\eta(\xi, \xi-x+y)]d\xi = \int_x^{x-y}\upsilon_1(\xi, \xi-x+y)d\xi. \quad (48)$$

Considering that $u_\xi(\xi,\ \xi-x+y) + u_\eta(\xi,\ \xi-x+y) = \frac{\partial}{\partial\xi}u(\xi, \xi-x+y)$, from equality (48), the solution to problem 5 is determined, representable in the form:

$$u(x,y) = \tau(x-y) - \int_x^{x-y}\upsilon_2(\xi, \xi-x+y)d\xi,\ (x,y) \in D_2. \quad (49)$$

Hence, the following theorem holds.

Theorem 1. Let the conditions be met (5), (6), (29), (32) and (45). Then the solution to Problem 1 exists and is unique. If condition (45) is not met, then the solution to Problem 1 in the domain D_1 suffers discontinuities of the first kind on the line $y = x$. Therefore, condition (45) provides a sufficient condition for the continuity of the desired solution along the line $y = x$.

Example 1. Let $\forall y \in [0, h]$: $\tilde{\varphi}_1(y) = \varphi_3(y) - \varphi_1'(y) \equiv 0$, $\tilde{\varphi}_2(y) = \varphi_4(y) - \varphi_2'(y) \equiv 0$, $\tilde{\psi}(x) = -\sqrt{2}\psi(x) \equiv 0$. Therefore, $\forall x \in [0, l]$: $\varphi_3(y) = \varphi_1'(y)$, $\varphi_4(y) = \varphi_2'(y)$, $\forall x \in [0, \frac{l}{2}]$: $\psi(x) = 0$. Then problem 2 has a trivial solution, that is $\forall (x,y) \in \bar{D}_1$: $\upsilon_1(x,y) \equiv 0$, $\forall (x,y) \in \bar{D}_2$: $\upsilon_2(x,y) \equiv 0$. From formula (44), the solution of Problem 1 is determined in the form:

$$u(x,y) = \begin{cases} \varphi_1(y-x), 0 \le x < y, 0 \le y \le h, \\ \varphi_2(l-x+y), y < x \le l, 0 \le y \le h. \end{cases} \quad (50)$$

It is obvious that if $\varphi_1(0) = \varphi_2(l)$, then $u(x,y) \in C(\bar{D}_1)$. It follows from (50) that $\tau(x) = \varphi_2(l-x)$. Solution of Problem 1 in the domain D_2 is represented by the formula (49):

$$u(x, y) = \tau(x - y) = \varphi_2(l - x + y),\ (x, y) \in D_2.$$

Thus, the existence and uniqueness of the solution to Problem 1 for a third-order equation have been proven, where a mixed parabolic-hyperbolic second-order operator is applied to a first-order linear differential operator. Various formulations of boundary value problems for a third-order equation with two independent variables were considered in the work of B.I. Islomov & O.Kh. Abdullaev [13], where the unique solvability of a nonlocal problem with an integral gluing condition for a third-order equation with a parabolic-hyperbolic operator was proven.

This operator includes the Caputo fractional derivative and a nonlinear term involving the trace of the solution $u(x, 0)$. Unlike this work, the equation under consideration is a third-order equation in which a first-order differential operator with coefficients a, b, and c acts on a second-order parabolic-hyperbolic operator.

A.N. Mironov [14] considered the third-order equation with a dominant partial derivative (Bianchi equations), providing the formulation of the Darboux problem and the definition of the Riemann–Hadamard function. Relying on the possibility of representing the Riemann function explicitly for one class of third-order Bianchi equations equivalent by function, sufficient conditions on the coefficients of the Bianchi equation were proposed to ensure the construction of the Riemann–Hadamard function in terms of hypergeometric functions.

In the work by Zh.A. Balkizov *et al*. [15], a boundary value problem with displacement was studied for a nonhomogeneous third-order parabolic-hyperbolic type equation, where one of the boundary conditions is given as a linear combination of the values of the sought function on independent characteristics. The work established necessary and sufficient conditions for the existence and uniqueness of a regular solution to the problem. It was shown that if the necessary conditions on the given functions found in the work are violated, the corresponding homogeneous problem has infinitely many linearly independent solutions, and the set of solutions to the corresponding nonhomogeneous problem may exist only under additional requirements on the given functions.

In the work of R.Kh. Makaova [16], a mixed boundary value problem was studied for a nonhomogeneous third-order hyperbolic-type Aller equation. Using the method of separation of variables, the theorem on existence and uniqueness of a regular solution was proven. A representation of the Green's function was obtained. The explicit form of the regular solution to the problem under study was written out.

A.N. Mironov & A.P. Volkov [17] proved the existence and uniqueness of the solution to a boundary value problem with conditions on one of the characteristics and on a free line for a system of hyperbolic equations with multiple characteristics. An analogue of the Riemann–Hadamard method was developed for the stated problem, and the definition of the Riemann–Hadamard matrix was given. The solution to the problem was constructed in terms of the introduced Riemann–Hadamard matrix.

A.K. Urinov & K.S. Khalilov [18] formulated and studied a nonclassical problem with an integral condition for a third-order parabolic-hyperbolic equation. The unique solvability of the posed problem was proven using the method of integral equations. Moreover, the posed problem was equivalently reduced to a problem for a second-order parabolic-hyperbolic equation with an unknown right-hand side. In the study of this latter problem, formulas for solving the Cauchy problem for a hyperbolic equation with a singular coefficient and a spectral parameter, as well as solutions of the first boundary value problem for the parabolic Fourier equation, were used.

In the article by O.Kh. Abdullaev & T.K. Yuldashev [19], the existence and uniqueness of solutions to inverse problems with a nonlinear gluing condition for a loaded parabolic-hyperbolic type equation were investigated. The problem was reduced to the study of a nonlinear Fredholm integral equation of the second kind. The theorem on existence and uniqueness of the solution was proved by the method of successive approximations.

O.M. Dzhokhadze *et al*. [20] studied a mixed problem with Dirichlet and Poincaré boundary conditions for a second-order hyperbolic equation and systems. In the linear case, an explicit representation of the solution was given, and questions of uniqueness and solvability of the posed problem were also investigated depending on the nature of the nonlinearities present in the system.

The work of A. Matchanova [21] is devoted to the solution of a local problem for a third-order parabolic-hyperbolic equation with a Caputo fractional derivative. The considered problem includes the third boundary condition in the parabolic domain and a continuity condition on the line $y = 0$. The existence of the solution was proven using the theory of Volterra-type integral equations.

A.I. Kozhanov & G.R. Ashurova [22] investigated the well-posedness of inverse problems of determining, together with the solution, a degenerate differential equation with multiple characteristics and an unknown coefficient defining the external influence (free term). The nature of degeneracy in the studied equation, as well as the form of the unknown coefficient, are determined by the time variable. For the studied problems, theorems on the existence and uniqueness of regular solutions having all generalised derivatives according to S.L. Sobolev entering the equation are proven.

M.G. Beshtokov [23] considered initial-boundary value problems for a fractional-order moisture transfer equation with a nonlocal linear source and variable coefficients. Assuming the existence of a regular solution for each of the considered first and third initial-boundary value problems, an a priori estimate in differential form was obtained. Estimates in difference form and convergence of the solution of each difference problem were obtained. Numerical calculations illustrating the obtained theoretical results were performed.

Yu.P. Apakov & A.A. Sopuev [24] proved the existence of a unique solution for nonlocal conjugation problems in a rectangular domain for a third-order partial differential equation, when for $y > 0$ the characteristic equation has three multiple roots, and for $y < 0$ it has one simple and two multiple roots. Using the Green function and the method of integral equations, the solution of the problems is equivalently reduced to the solution of a boundary value problem for the trace of the sought function at $y = 0$, and then to the solution of a Fredholm integral equation of the second kind, the solvability of which is proved by the method of successive approximations. The solution of the problem for $y > 0$ is constructed by the Green function method, and for $y < 0$ – by reducing the problem to a two-dimensional Volterra integral equation of the second kind.

It should be noted that the cited works considered various formulations of boundary value problems for a third-order equation with two independent variables. In the present work, one of the variants of the boundary value problem formulation for a third-order equation was studied, where a mixed parabolic-hyperbolic second-order operator with a line of type change is involved $y = 0$, is applied to a first-order linear differential operator.

Conclusions

The unique solvability of the main problem 1 for equation (1) is proved when the mixed parabolic-hyperbolic operator is applied on the left to a first-order differential operator. In solving problem 1, the method of reducing the order of the equation was used, as a result of which this problem is reduced to problem 2 (an analogue of the Tricomi problem) for a second-order mixed parabolic-hyperbolic equation with a line of type change $y = 0$. The peculiarity of this problem is that this line is a characteristic of equation (1) in the domain D_1. Using the methods of integral equations, Green's function, and Riemann function, the existence and uniqueness of the solution to problem 2, when the equation in the domain D_1 belongs to the parabolic type, and in the domain D_2 – of hyperbolic type. A key point in solving Problem 2 was finding the trace of the sought function and its derivative on the line $y = 0$. Next, the solution of problem 2 was decomposed into solutions of the first boundary value problem for equation (13) in the domain D_1, and in the domain D_2 – to the solution of the Cauchy problem for the hyperbolic type equation (14). Then, problems 4 and 5 were considered. The peculiarity of the solution to problem 4 was that on the line $y = x$ the domain D_1. The sought solution undergoes a first kind discontinuity. Therefore, it is additionally required to satisfy the continuity condition of the sought solution along the line $y = x$, as a result, the continuity of the function $u(x, y)$, as the solution of problem 4 in the domain D_1 is provided. When the coefficients at the lower-order terms of the equation are absent, the continuity of the solution is ensured by the fulfillment of the compatibility condition: $\varphi_1(0) = \varphi_2(l)$. Thus, the set goal in solving the main problem 1 has been achieved. The indicated research method is also applicable if the gluing conditions have first-kind discontinuities on the line of type change.

It should also be noted that there is a need to study, in the direction of this research topic, the case when the line of change of the type of the equation is a line $x = 0$ since in this case, unlike in the present study, this line is not a characteristic of either the parabolic or the hyperbolic equation.

Acknowledgements

The authors express their gratitude to Doctors of Physical and Mathematical Sciences, Professors K.B. Sabitov and Yu.P. Apakov, for their valuable advice and suggestions.

Funding

None.

Conflict of Interest

None.

References

[1] Belakroum K. On the stability of nonlocal boundary value problem for a third order PDE. IJAM. 2021;34(2):391–400. DOI: 10.12732/ijam.v34i2.14

[2] Ashurov RR, Fayziev YuE. Uniqueness and existence for inverse problem of deter mining an order of time-fractional derivative of subdiffusion equation. Lobachevskii J Math. 2021;42(3):508–16. DOI: 10.1134/S1995080221030069

[3] Balkizov ZhA. Boundary-value problem with shift for a third-order parabolic-hyperbolic equation, results of science and technology. Mod Math Its Appl Themat Rev. 2021;198:33–40. DOI: 10.36535/0233-6723-2021-198-33-40

[4] Ochilova NK, Yuldashev TK. On a nonlocalboundary value problem for a degenerate parabolic-hyperbolic equation with fractional derivative. Lobachevskii J Math. 2022;43(1):229–36. DOI: 10.1134/S1995080222040175

[5] Kozhanov AI, Tarasova GI. The Samarskii–Ionkin problem with integral perturbation for a pseudoparabolic equation. Bull Irkutsk State Univ Ser Math. 2022;42:59–74. DOI: 10.26516/1997-7670.2022.42.59

[6] Durdieva DK, Merajova ShB. Inverse problem for an equation of mixed parabolic-hyperbolic type with a Bessel operator. Sib J Ind Math. 2022;25(3):14–24. DOI: 10.33048/SIBJIM.2022.25.302
[7] Sidorov SN. Three-dimensional initial boundary value problem for a parabolic-hyperbolic equation with a degenerate parabolic part. News Univ Math. 2023;(4):51–64. DOI: 10.26907/0021-3446-2023-4-51-64
[8] Makaova RKh. On a mixed problem for a third order degenerating hyperbolic equation. Bull KRASEC Phys Math Sci. 2023;44(3):19–29. DOI: 10.26117/2079-6641-2023-44-3-19-29
[9] Durdiev DK. Inverse problem for an equation of mixed parabolic-hyperbolic type with a characteristic line of change. Bull Samara State Tech Univ Ser Phys Math Sci. 2023;27(4):607–20. DOI: 10.14498/vsgtu2027
[10] Sabitov KB. Direct and inverse problems for equations of mixed parabolic-hyperbolic type. Moscow: Nauka; 2016. 272 P.
[11] Krasnov ML, Kiselev AI, Makarenko GI. Integral equations: Problems and examples with detailed solutions. Moscow: Yeditorial URSS; 2003. 194 P.
[12] Polyanin AD. Handbook of linear equations of mathematical physics. Moscow: Fizmatlit; 2001. 576 P.
[13] Islomov BI, Abdullaev OKh. On non-local problems for third order equation with Caputo operator and non-linear loaded part. Ufa Math. J. 2021;13(3):44–56. DOI: 10.13108/2021-13-3-44
[14] Mironov AN. Construction of the Riemann-Hadamard function for the three-dimensional Bianchi equation. Izv Univ Math. 2021;(3):76–82. DOI: 10.26907/0021-3446-2021-3-76-82
[15] Balkizov ZhA, Ezaova AG, Kanukoeva LV. Boundary value problem with displacement for a third-order parabolic-hyperbolic equation. Vladikavkaz Math J. 2021;23(2):5–18. DOI: 10.46698/d3710-0726-7542-i
[16] Makaova RKh. About one mixed problem for the inhomogeneous Hallaire equation. Adyghe Int Sci J. 2022;22(2):29–33. DOI: 10.47928/1726-9946-2022-22-2-29-33
[17] Mironov AN, Volkov AP. On the Darboux problem for a hyperbolic system of equations with multiple characteristics. Russ Math (Iz VUZ). 2022;66(8):31–6. DOI: 10.26907/0021-3446-2022-8-39-45
[18] Urinov AK, Khalilov KS. A nonlocal problem for a third order parabolic-hyperbolic equation with a singular coefficient. J Sib Fed Univ Math Phys. 2022;15(4):467–81. DOI: 10.17516/1997-1397-2022-15-4-467-481
[19] Abdullaev OKh, Yuldashev TK. Inverse problems for the loaded parabolic hyperbolic equation involves Riemann–Liouville operator. Lobachevskii J Math. 2023;44(3):1080–90. DOI: 10.1134/S1995080223030034
[20] Dzhokhadze OM, Kharibegashvili SS, Shavlakadze NN. Mixed problem for one class of nonlinear hyperbolic systems of the second order with Dirichlet and Poincaré boundary conditions. Math Notes. 2023;114(5): 748–62. DOI: 10.1134/S000143462311010X
[21] Matchanova A. On a problem for the third-order equation of parabolic-hyperbolic type with the Caputo operator. Bull Inst Math. 2023;6(1):78–86.
[22] Kozhanov AI, Ashurova GR. Third-order differential equations with multiple characteristics: Degeneration and unknown external influence. Chelyabinsk Phys Math J. 2024;9(4):585–95. DOI: 10.47475/2500-0101-2024-9-4-585-595
[23] Beshtokov M.Kh. Initial-boundary value problems for the moisture transfer equation with fractional derivatives of different orders and a non-local linear source. Vladikavkaz Math J. 2024;26(3):5–23. DOI: 10.46698/l0699-2536-6844-a
[24] Apakov YuP, Sopuev AA. Nonlocal problems for a mixed parabolic-hyperbolic equation of the third order. Chelyabinsk Phys Math J. 2025;10(1):5–16. DOI: 10.47475/2500-0101-2025-10-1-5-16

Үчүнчү тартиптеги аралаш типтеги өзгөрмөлүү коэффициенттери бар теңдеме үчүн чек аралык маселелер

Адахимжан Сопуев
Физика-математика илимдеринин доктору, профессору
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0009-0008-8530-6629

Бактыбек Нуранов
Улук окутуучусу
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0009-0003-4120-8780

Аннотация. Төмөнкү мүчөлөрү өзгөрмөлүү коэффициенттер болгон үчүнчү тартиптеги аралаш типтеги теңдеме үчүн функциянын өзүнүн жана анын биринчи жана экинчи тартиптеги туундулары үчүн теңдеменин түрү өзгөрө турган $y = 0$ сызыгында жабыштыруу шарттары орун алган учурдагы чек аралык маселенин чечиминин бар экендиги жана жалгыздыгы далилденген. Бул маселе экинчи тартиптеги аралаш параболалык-гиперболалык оператор биринчи тартиптеги турактуу коэффициенттүү сызыктуу дифференциалдык операторго колдонулган учурда каралат. Теңдеменин тартибин төмөндөтүү жолу менен каралып жаткан маселе функциянын өзүн жана анын биринчи тартиптеги туундусун теңдеменин түрү өзгөрө турган $y = 0$ сызыгында жабыштыруу шарттары менен экинчи тартиптеги аралаш параболалык-гиперболалык теңдеме үчүн Трикоми маселесине келтирилет. Областтардын параболалык жана гиперболалык бөлүктөрүнөн алынган теңдемелер системасын жоюу ыкмасы менен маселенин чечилиши экинчи түрдөгү Фредгольмдун интегралдык теңдемесинин чечилүүчүлүгүнө келтирилет. Интегралдык теңдеменин чечилиши үчүн жетиштүү шарт теңдеменин ядросун баалоо аркылуу алынат. Маселени чечүү каралып жаткан аймактарда эки маселеге бөлүнөт: аймактын параболалык бөлүгүндө жылуулук теңдемеси үчүн биринчи чек аралык маселе Грин функциясы ыкмасы менен чыгарылат, ал эми теңдеменин мүнөздөөчү сызыктары жана $y = 0$ сызыгы менен чектелген аймакта гиперболалык теңдеме үчүн Римандын функциясын куруу ыкмасы менен маселени чечүү Коши маселесин чечүүгө алып келинет. Ийри сызыктуу интегралды колдонуу менен каралып жаткан аймактардагы маселенин чечими табылган. Теңдеменин түрүнүн $y = 0$ өзгөрүү сызыгында функциянын өзүнүн жана анын биринчи жана экинчи тартиптеги туундуларынын үзгүлтүксүздүгүн талап кылуу зарылчылыгы негизделген. Чек аралык маселесинин классикалык чечиминин жашашы үчүн жетиштүү шарттар табылган. Краевая маселенин чечилиш шарттары аэрогидродинамика, геофизика жана инженердик жылуулук техникасы тармагындагы колдонмолук маселелерди сандык ыкмалар менен чечүүнүн теориялык негизин түзөт

Негизги сөздөр: чечимдин жашашы; чечимдин жалгыздыгы; Гриндин функциясы; Римандын функциясы; интегралдык тендеме; тартибин төмөндөтүү методу

Краевые задачи для уравнения смешанного типа 3-го порядка с переменными коэффициентами

Адахимжан Сопуев
Доктор физико-математических наук, профессор
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0009-0008-8530-6629

Бактыбек Нуранов
Старший преподаватель
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0009-0003-4120-8780

Аннотация. Доказывается существование и единственность решения краевой задачи для уравнения смешанного типа третьего порядка с переменными коэффициентами при младших членах с условиями склеивания самой функции и её производных первого и второго порядков на линии $y = 0$ изменения типа уравнения, когда смешанный параболо-гиперболический оператор второго порядка применяется к линейному дифференциальному оператору первого порядка с постоянными коэффициентами. Методом понижения порядка уравнения задача сводилась к задаче Трикоми для уравнения смешанного параболо-гиперболического типа второго порядка с непрерывными условиями склеивания самой функции и её производной первого порядка по y на линии изменения типа уравнения. Методом исключения системы уравнений, полученных из параболической и гиперболической части областей, разрешимость задачи сводилась к разрешимости интегрального уравнения Фредгольма второго рода. Получено достаточное условие разрешимости интегрального уравнения через оценки ядра уравнения. Решение задачи расщеплялось на две задачи в рассматриваемых областях: в параболической части области методом функции Грина решалась первая краевая задача для уравнения теплопроводности, а в гиперболической части области, ограниченная характеристиками уравнения и линией $y = 0$, решение задачи методом построения функции Римана определялась как решение задачи Коши. Применением криволинейного интеграла найдено решение задачи в рассматриваемых областях. Обосновывалась необходимость требования непрерывности самой функции и её первых двух производных по y на линии изменения типа уравнения. Установлены достаточные условия для однозначной классической разрешимости краевой задачи. Полученные условия разрешимости краевой задачи обеспечили теоретическую основу для разработки численных методов решения прикладных задач в аэрогидродинамике, геофизике и инженерной теплотехнике

Ключевые слова: существование; единственность; функция Грина; функция Римана; интегральное уравнение; метод понижения порядка

VOL. 4, No. 1
Journal of Osh State University.
Mathematics. Physics. Technical Sciences
2025
Journal of Osh State University
Journal homepage: https://mpht-oshsu.com/en

UDC 517.928
DOI: 10.52754/16948645_2025_4(1)_41

Article's History:
Received 03.02.2025 Revised 20.05.2025 Accepted 19.06.2025

The influence of small perturbation on phenomenon of delayed loss of stability

Abdilaziz Akmatov*
Senior Lecturer
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0009-0004-6756-4262

Kylymgul Mamadjanova
Master's Student
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0009-0009-3525-9438

Arafatkan Baimamatova
Master's Student
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0009-0009-2966-4496

Eliza Islamidin kyzy
Master's Student
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0009-0002-2879-5071

Abstract. The study of solutions to singularly perturbed problems remains relevant, as many mathematical models in technical and natural sciences are described by such differential equations. Despite existing research, there remains a need for a more in-depth analysis and further study of the in-fluence of small perturbations on the phenomenon of delayed loss of stability. The aim of this study was to examine the influence of a small perturbation on the phenomenon of delayed loss of stability, as well as to justify the limit transition confirming the convergence of solutions of the perturbed and unperturbed problems. To achieve this goal, analytical methods were employed, including the level lines method and methods for selecting descending integration paths, which made it possible to rigorously substantiate the limit transitions between the perturbed and unper-turbed problems. The study established that in the absence of a small perturbation, the phenomenon of delayed loss of stability persists regardless of the location of the zeros of the eigenvalues whether on the real axis or in the complex plane. In the presence of a small perturbation, the situation changes: if the eigenvalues have zeros on the real axis, the delay phenomenon does not occur. However, if the zeros are located in the complex plane, the delay is observed only over a limited time interval. In the case where the eigenvalues have poles, the small perturbation does not affect the presence of the phenomenon persists in all cases. Thus, the influence of a small perturbation on the delayed loss of stability depends significantly on the nature of the eigenvalues. It was also substantiated that, under certain conditions on the small perturbation, convergence of solutions is ensured when transitioning from the perturbed problem to the unperturbed one. The results of the study provide a justification for the existence and nature of delayed loss of stability in broader functional spaces, which is important for applied problems in modelling unstable processes

Keywords: small parameter; limit transition; eigenvalues; stability of solutions; integral curves

Suggested Citation:
Akmatov A, Mamadjanova K, Baimamatova A, Islamidin E. The influence of small perturbation on phenomenon of delayed loss of stability. J Osh State Univ Math Phys Tech Sci. 2025;4(1):41–9. DOI: 10.52754/16948645_2025_4(1)_41

*Corresponding author

Introduction

The phenomenon of delayed loss of stability plays an important role in the theory of singularly perturbed first-order differential equations during changes in stability conditions. The stable and unstable intervals of the solution to singularly perturbed first-order differential equations are determined by the real parts of the eigenvalues. The identification of the negative interval of the real part of the eigenvalue integral justifies the emergence of the phenomenon of delayed loss of stability. Such a case occurs when the zeros of the eigenvalues lie in the complex plane. When they lie on the real line, the question arises about the presence of the extended loss of stability effect and the nature of the system's response to small perturbations. Given the insufficient degree of study of such cases, the presented research is of high relevance.

When the limit equation has turning points, the work of A.G. Eliseev [1] studied the construction of an asymptotic solution of the linear Cauchy problem with a weak turning point of the limit operator using the Lomov regularisation method. The main singularities of this problem are presented in explicit form. Estimates in terms of ε are provided, describing the behaviour of the singularities as $\varepsilon \to 0$. Asymptotic convergence of the regularised series has been proven. The obtained results are illustrated by an example. Small perturbations and the case of stability change are not considered in this work.

In the work of P. Kaklamanos *et al*. [2], an autonomous singularly perturbed system with two fast and one slow variable was considered, in which the linearisation of the fast variable subsystems has intersecting or closely spaced eigenvalues. It has been shown that such a spectral structure leads to the emergence of the delayed loss of stability effect, where the system's trajectory remains near an unstable equilibrium longer than expected. The authors proposed generalised entry-exit relations formulas that allow for a quantitative description of the nature and duration of this delayed transition, including the influence of eigenvalue crossings on the geometry of the phase space.

Sometimes, when the zeros of the eigenvalues lie on the real line, it is possible to determine stable and unstable intervals, but the integral taken over the eigenvalues does not allow for the determination of a stable interval. In this case, the point of stability condition change is chosen as the initial point, which allows for the determination of a stable region for conducting the study. A.A. Akmatov *et al.* [3] noted this case in their work.

S.K. Karimov *et al*. [4] studied the case when the zeros of the eigenvalues lie on the real axis and small perturbations are present. Here, the eigenvalues satisfy all the conditions for determining stable and unstable intervals. Moreover, on the complex plane, the lines passing through the points of stability change divide the region into four parts. The choice of a descending integration path connecting the initial and final points within the solution domain of the problem is accompanied by certain difficulties. These difficulties are resolved by means of parallel lines, which are well known in projective geometry. However, in this case, there are certain nuances that require explanation. Moreover, it does not fully cover the influence of a small perturbation on the solution of the singular problem.

In the work of A.G. Eliseev [5], based on the regularisation method of S.A. Lomov, an asymptotic solution of the singularly perturbed Cauchy problem for a parabolic equation with the presence of a strong turning point is constructed. The regularisation method allows obtaining a uniform asymptotic solution of the problem on the entire real axis. The idea of the work is based on previously developed methods for solving the singularly perturbed Cauchy problem in the case of a simple turning point of the limiting operator with a natural exponent. However, the case of a singularly perturbed problem with a small perturbation is not considered in this work.

It is known that the zeros of eigenvalues located on the complex plane determine the delay time and the occurrence of the phenomenon of delayed loss of stability. A similar effect was also considered in the context of the FitzHugh-Nagumo model, where a delay in the transition from the steady state to the oscillatory regime is observed during a slow passage through the Hopf bifurcation point. As shown in one of the studies by S.M. Baer *et al*. [6], the system enters oscillations at a parameter value significantly exceeding the critical one, indicating the presence of memory and delay effects not accounted for in the classical bifurcation analysis.

M.N. Nurmatova [7] studied the case when the eigenvalues were complex conjugates, which means that the zeros of the eigenvalues belong to the complex plane. One of the features of the study is the change in the zeros of the eigenvalues, which, in turn, leads to a change in the delay time. Only the smallest zeros of the eigenvalues affect the obtained estimate. A small perturbation is present, and the eigenvalues are complex conjugates, which indicates the existence of the phenomenon of delayed loss of stability.

The aim of the study was to investigate the influence of a small perturbation on the behaviour of the solutions of the problem, as well as to examine its effect on the phenomenon of delayed loss of stability. An important aspect of the work was the justification of the limiting transition, which confirmed the convergence of the solutions of the perturbed and unperturbed problems.

Materials and Methods

Within the framework of the study, the problem of analysing the characteristics of solutions to singularly perturbed differential equations of the following form was considered:

$$\varepsilon x'(t,\varepsilon)=a(t)x(t,\varepsilon)+\varepsilon[f(t)+g(t,x)], \quad (1)$$

$$x(0,\varepsilon)=x^0(\varepsilon),\ |x^0(\varepsilon)|=O(\varepsilon), \quad (2)$$

where $0<\varepsilon$ – is a small parameter, $g(t,x(t,\varepsilon))$ – is an analytic function of two variables, $t\in T$ is a finite or infinite domain. For example, $g(t,x(t,\varepsilon))$ may turn out to be polynomials in the variable x with analytic coefficients on the domain T.

Definition. The expression $\varepsilon f(t)$, where $0<\varepsilon$ is a small parameter, is called a small perturbation.

The unperturbed equation $a(t)\ \tilde{x}(t,\ 0)=0$ has the zero solution $\tilde{x}(t)=0$. In the course of the study, it is necessary to prove the limiting equality:

$$\lim_{\varepsilon\to 0} x(t,\varepsilon)=\tilde{x}(t), \text{ as } t\in T. \quad (3)$$

Let the following conditions be satisfied:

I) $\varepsilon f(t)=0$ and $a(t)=\alpha(t)$ or $a(t)=\alpha(t)+i\beta(t)$.
II) $\varepsilon f(t)\neq 0$ and $a(t)=\alpha(t)$ or $a(t)=\alpha(t)+i\beta(t)$.

Here $a(t)=\alpha(t)$, defines the stable and unstable intervals of the eigenvalues, indicating the transition point from stability to instability. By analysing the real part of the eigenvalue $a(t)=\alpha(t)+i\beta(t)$ and solving it as an equation, one can determine the stable and unstable intervals, as well as the transition points from stability to instability.

To analyse the stable interval on the numerical axis, the method of integration over eigenvalues, previously determined on stable and unstable intervals, was applied. This point was one of the main conditions of the problem solution study. At the next stage, the method of Lagrange variation was applied to obtain an analytical representation of the solutions of problems (1) and (2). This method allowed expressing the solution in an integral form suitable for subsequent estimation and theoretical analysis within the framework of the posed problem. To estimate the solution presented in analytical form, a descending integration method was chosen. Accordingly, the level set method [8] or the stationary phase method [9], or simply an appropriate descending integration method, were used. Within the framework of the study of boundary layers, the method [10] related to the investigation of transition curves was applied.

The obtained analytical expression was solved using the method of successive approximations, traditionally applied in the theory of differential equations. The choice of this method was determined by its ability to provide the construction of an approximate solution with the subsequent possibility of obtaining an exact estimate. Majorant series were used to demonstrate the convergence of the obtained estimates. This, in turn, facilitated the process of achieving the corresponding convergence of the estimates.

At the next stage, it was planned to prove the uniqueness of the solution by the method of contradiction. However, this step was not carried out within the framework of the work, as it was reduced to standard formal procedures. Nevertheless, the structure of the study was maintained in accordance with the generally accepted approach.

Results and Discussion

Let condition I be satisfied. However, its fulfillment did not eliminate the necessity of solving the differential equation using the Lagrange method, since the problems defined by equalities (1) and (2) were nonlinear. Thus, in accordance with the Lagrange method, a homogeneous approximation was identified, corresponding to the singularly perturbed first-order ordinary differential equation given in equality (1). Then:

$$\varepsilon\frac{dx(t,\varepsilon)}{dt}=a(t)x(t,\varepsilon).$$

By separating the variables and integrating, the general solution was obtained: $x(t,\varepsilon)=C\,exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t}a(s)ds\right)$, where C – is an arbitrary constant. Now, by varying C, taking the value $C=C(t)$. As a result, the following was obtained:

$$x(t,\varepsilon)=C(t)\,exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t}a(s)ds\right).$$

Substituting the derivative $x'(t,\varepsilon)$ and the function $x(t,\varepsilon)$ itself into equality (1), performing some transformations, the following was obtained:

$$x(t,\varepsilon)=x^0(\varepsilon)\times$$
$$\times\,exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t}a(s)ds\right)+\int_{t_0}^{t}g(\tau,x(\tau,\varepsilon))\,exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t}a(s)ds\right)d\tau. \quad (4)$$

The integral problem defined by equality (4) was solved using the method of successive approximations. This method simplifies calculations, takes into account small perturbations, and constructs an asymptotic approximation of the perturbed problem's solution to the solution of the corresponding unperturbed problem. The successive approximations were defined as follows:

$$x_0(t,\varepsilon)=0,$$
$$x_1(t,\varepsilon)=x^0(\varepsilon)\,exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t}a(s)ds\right),$$
$$x_2(t,\varepsilon)=x^0(\varepsilon)\,exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t}a(s)ds\right)+$$
$$+\int_{t_0}^{t}g(\tau,x_1(\tau,\varepsilon))\,exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t}a(s)ds\right)d\tau,$$
$$\dots\dots\dots,$$
$$x_n(t,\varepsilon)=x^0(\varepsilon)\,exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t}a(s)ds\right)+$$
$$+\int_{t_0}^{t}g(\tau,x_{n-1}(\tau,\varepsilon))\,exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t}a(s)ds\right)d\tau,$$
$$(n=0,\ 1,\ 2,\dots).$$

The unperturbed equation had the zero solution $\tilde{x}(t)=0$. The initial point belonged to the stable segment, therefore, according to the stability criterion, was chosen $|x(t_0,\varepsilon)|=|x^0(\varepsilon)|=O(\varepsilon)$. The study was conducted in the complex plane. Then:

$$u(t_1, t_2) = Re\int_{t_0}^{t_1+it_2} a(s)ds,$$
$$\vartheta(t_1, t_2) = Im\int_{t_0}^{t_1+it_2} a(s)ds,$$

where $t_1, t_2 \in R$. The solutions of problems (1) and (2) were studied in the domain $T = \{(t_1; t_2): u(t_1, t_2) \le 0\}$, which was defined by the function $u(t_1, t_2)$. From equality (4), the following notation was introduced:

$$A(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right)$$
and $I(t,\varepsilon) = \int_{t_0}^{t} g(\tau, x(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right)d\tau.$

In the domain T, the estimate of the absolute value $|A(t,\varepsilon)| = \left|x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right)\right| = O(\varepsilon)$, turned out to be valid. This estimate held due to the absolute value of the initial problem $|x(t_0, \varepsilon)| = |x_0(\varepsilon)| = O(\varepsilon)$.

The zeros of the function defined by the integral of the eigenvalues in the complex plane determined the lines. These lines, in turn, marked the boundaries of the domains under study in the complex plane, within which problems (1) and (2) were solved. A boundary layer was formed in a sufficiently small neighbourhood of these boundaries. In this layer, the estimates defined by the function $A(t, \varepsilon)$ became significant. However, since the initial point was chosen within a stable interval, taking into account the condition $|x^0(\varepsilon)| = O(\varepsilon)$, an estimate could be obtained.

To estimate the function $I(t, \varepsilon)$, integration paths were chosen in the domain T. In order for the successive approximations to remain bounded, it was necessary to satisfy the condition $u(t_1, t_2) - u(\tau_1, \tau_2) \le 0$, that is, the integration paths had to be descending from the initial point to the final one. As a result, the first approximation was determined by the estimate $|A(t, \varepsilon)|$, and then for all the $(t_1; t_2) \in T$ following estimate held:

$$|x_1(t, \varepsilon)| \le C\delta(\varepsilon), \tag{5}$$

where C is an arbitrary constant.

The function $I(t, \varepsilon)$ had values except within the boundary layers. Therefore, it was necessary to follow a descending path of integration. For $\delta(\varepsilon)$, the statement held true $\varepsilon = o(\delta(\varepsilon))$, provided that ε tended to zero. The remaining approximations were determined by the estimate of the function $I(t, \varepsilon)$ in the domain T. A majorant expression in the form of a series was constructed, corresponding to the difference:

$$\sum_{n=1}^{\infty}[x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)]. \tag{6}$$

The convergence of successive approximations was studied. Subsequently, positive constants that did not play a significant role in the reasoning were denoted by the same letter C. It is assumed that the following condition was satisfied:

$$|g(t, x) - g(t, \tilde{x})| \le \beta|x - \tilde{x}|, \tag{7}$$

where $0 < \beta$ – is a certain constant. Then:

$$x_2(t,\varepsilon) - x_1(t,\varepsilon) = \int_l [g(\tau, x_1(\tau,\varepsilon))]\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right)d\tau.$$

The absolute value was determined as follows:

$$|x_2(t,\varepsilon) - x_1(t,\varepsilon)| \le \beta \int_l |x_1(t,\varepsilon)|\, exp\left(\frac{u(t_1,t_2) - u(\tau_1,\tau_2)}{\varepsilon}\right)\cdot$$
$$\cdot\, |d\tau| \le \beta C\sqrt{\varepsilon}\int_l exp\left(\frac{u(t_1,t_2) - u(\tau_1,\tau_2)}{\varepsilon}\right)\cdot |d\tau| \le (C\delta(\varepsilon))^2.$$

The corresponding estimate was obtained

$$|x_2(t, \varepsilon) - x_1(t, \varepsilon)| \le (C_\delta(\varepsilon))^2. \tag{8}$$

The following expression then took place:

$$|x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)| \le (C_\delta(\varepsilon))^n, (n = 1, 2,...).$$

The proof of the uniqueness of the solution was carried out similarly to that performed in the work of L.S. Pontryagin & E.F. Mishchenko [11]. The following theorem holds true.

Theorem 1. In the domain T, the problem (1), (2) has a unique solution $x(t, \varepsilon)$, representable in the form $x(t, \varepsilon) = \sum_{n=1}^{\infty}[x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)]$, and on T the estimate $|x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)| \le (C_\delta(\varepsilon))^n$, $(n = 1, 2,...)$ holds, where $0 < C$ is a constant number.

Let condition II be satisfied. In this case, the choice of the initial point, the determination of stable and unstable intervals, as well as the definition of the domain T for solving problem (1), (2), were carried out in full analogy with case I. The problem (1), (2) in this case was also solved using the Lagrange variation method. Without repeating the steps carried out in section I, by incorporating the features of this case, when the function $f(t)$ was nonzero, the solution could be written as follows:

$$x(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t}[f(\tau) + g(\tau, x)]\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right)d\tau. \tag{9}$$

The problem posed in (9) was again solved using the method of successive approximations. The successive approximation was written for the case when the function $f(t)$ was nonzero. Thus, the following expression was obtained:

$$x_0(t, \varepsilon) = 0,$$
$$x_1(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t} f(\tau)\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right)d\tau,$$
$$x_2(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t} g(\tau, x_1(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right)d\tau,$$
$$\dots\dots\dots,$$
$$x_n(t,\varepsilon) = x^0(\varepsilon)\, exp\left(\frac{1}{\varepsilon}\int_{t_0}^{t} a(s)ds\right) +$$
$$+ \int_{t_0}^{t} g(\tau, x_{n-1}(\tau,\varepsilon))\, exp\left(\frac{1}{\varepsilon}\int_{\tau}^{t} a(s)ds\right)d\tau,$$
$$(n = 0, 1, 2,...).$$

In this case, since the function $f(t)$, which defines the nonhomogeneous part, was nonzero, the integration path at the first approximation was chosen so that it decreased from the initial point to the endpoint t. As a result, a value was obtained that represented an infinitesimal of a lower order compared to the estimate given in Theorem 1 and depended on the special points of the eigenvalues. Thus, in the domain T, the following estimate was obtained:

$$|x_1(t,\varepsilon)| \leq C\gamma(\varepsilon), \tag{10}$$

where C is an arbitrary constant. As can be seen, the estimate $\gamma(\varepsilon)$, defined by equation (10), tended to zero as the small parameter approached zero. However, the order of this convergence was lower than the order of the estimate in Theorem 1. This is because in that case, the first approximation was determined by the function $A(t, \varepsilon)$, whereas here it was determined by the function $x(t, \varepsilon)$. Since the solution $x(t, \varepsilon)$ included a nonhomogeneous part, the special points of the eigenvalues influenced the order of the solution estimate. For this reason, the equality $\varepsilon = o(\gamma(\varepsilon))$ held true as the small parameter tended to zero.

To prove the convergence of the estimates obtained through successive approximations, the majorant series (6) was also used in this case. Accordingly, assuming the fulfillment of the equality condition (7), and generally taking absolute values, the following estimate was obtained:

$$|x_n(t,\varepsilon) - x_{n-1}(t,\varepsilon)| \leq (C\gamma(\varepsilon))^n, \ (n = 1, 2,...).$$

The uniqueness of the solution was proved similarly to Theorem 1. The following theorem holds.

Theorem 2. In the domain T, the problem (1), (2) had a unique solution $x(t, \varepsilon)$ representable in the form $x(t,\varepsilon) = \sum_{n=1}^{\infty}[x_n(t,\varepsilon) - x_{n-1}(t,\varepsilon)]$, and the estimate $|x_n(t, \varepsilon) - x_{n-1}(t, \varepsilon)| \leq (C\gamma(\varepsilon))^n$, $(n = 1, 2,...)$ was valid on T, where $0 < C$ is a constant number.

The first theorem was studied in the absence of small perturbations and proved that the phenomenon of delayed loss of stability occurred independently of the singular points of the eigenvalues. The second theorem proved that this phenomenon took place when the singular points of the eigenvalues were located in the complex plane.

In their work, the authors S.F. Iglesias & S. Mirrahimi [12] studied the asymptotic behaviour of solutions of a Lotka-Volterra type parabolic equation with a periodically varying growth coefficient and nonlocal competition. It has been proven that, at large times, the solution converges to a unique periodic solution. At small mutations, the solution concentrates in the form of a single delta function, and the population size changes periodically over time. Moreover, using methods from the theory of Hamilton-Jacobi equations with constraints, a detailed asymptotic characterisation of such behaviour was obtained. For small but nonzero values of mutations, formal approximations of the moments of the population distribution were proposed, which makes it possible to describe the dynamics of its evolution more accurately. The authors also demonstrated how the obtained results can be applied to interpret and predict biological experiments, confirming the significance of mathematical modelling in studying the adaptation of populations in changing environments.

The authors D.A. Tursunov & G.A. Omaralieva [13] considered the Cauchy problem for a first-order ordinary differential equation with a small parameter at the derivative and a singularity at the initial point. A sufficient condition was found, the fulfillment of which leads to the appearance of an intermediate boundary layer in the singularly perturbed problem described by first-order ordinary differential equations. Using the modified boundary function method, a complete asymptotic expansion of the solution in the form of a series in the Erdélyi sense was constructed. The obtained expansion was justified, that is, an appropriate estimate was obtained for the remainder term. The study is devoted to the investigation of the boundary layer structure, for which necessary estimates were also obtained.

In the work of A.S. Ryabenko [14], a study was conducted on problems of evolutionary differential equations with a complex parameter. Of special interest was the behaviour of their solutions for large values of time, as it demonstrated their evolution. A homogeneous ordinary differential equation with a variable coefficient and a complex parameter was considered, to the study of which a wide class of problems of evolutionary differential equations could be reduced. Unlike the considered work, the parameter was complex and was not associated with the highest derivative, which affected the structure and behaviour of the solutions of this differential equation.

In the article by V.I. Uskov [15], the Cauchy problem for a first-order differential equation in a Banach space was considered, containing a small parameter at the highest derivative and an operator term of Fredholm type on the right-hand side. The relevance of the problem associated with a small parameter at the highest derivative is due to the need to model various physical processes, such as the behaviour of viscous flow, deformations of thin plates and shells, as well as supersonic flow around blunt bodies. The existence of a boundary layer has been identified in the work, which significantly affects the solution even under small perturbations. The asymptotic expansion of the solution is constructed using the Vasilieva-Vishik-Lyusternik method in the form of a power series in the small parameter, and its validity is confirmed by mathematical justification. The regular part of the decomposition is formed by means of the equation decomposition method, which involves a sequential reduction of the problem's dimensionality.

K.G. Kozhobekov *et al*. [16] constructed a uniform asymptotic expansion of the solution to the first boundary value problem for a singularly perturbed second-order parabolic equation. The Vishik-Lusternik method is used, as well as the maximum principle and methods of integrating ordinary differential equations. The solution is presented as the sum of the outer solution and several boundary layers that exponentially decay outside these layers. The remainder term is estimated, which confirms the asymptotic nature of the expansion.

The authors J. Penalva *et al*. [17] studied the phenomenon of delay of loss of stability in an autonomous fast-slow system with a piecewise-linear structure and a slowly varying parameter. It is shown that when the eigenvalues of the fast variables cross or approach each other, a delayed Hopf bifurcation occurs, known as the "way-in/way-out" effect. The authors presented the conditions for the occurrence of this phenomenon and described the so-called entry-exit functions, which depend on the initial conditions and the duration of the delay. Applied to a neuronal model (elliptic bursting), it is shown that these mechanisms are robustly realised even in simple piecewise-linear systems.

In the work by J.T. Zhusubaliyev *et al*. [18], a study was conducted on the bifurcation structure related to the stability of oscillations, bistability, and synchronisation of forced oscillations in a relay system with hysteresis. The behaviour of this system was described by a non-autonomous differential equation with a discontinuous right-hand side. The basic properties of this equation were considered first. Then, a method for obtaining the first return map from this vector field was presented, and it was shown that, depending on the parameter values, such a map can be either a diffeomorphism of the circle or a map with discontinuities. An equation has been identified that divides the parameter plane into regions where the map is either smooth and invertible or discontinuous. A detailed analytical and numerical bifurcation analysis has been carried out, explaining the mechanism of transition between stable capture regimes, bistable states, and chaotic dynamics. Moreover, this work allows the system to be considered as a mathematical model of an oscillatory process describing the dynamics of transitions between different operating modes of the relay system.

In their work, S. Karimov & G.M. Anarbaeva [19] investigated the solution of a singularly perturbed problem under changing stability conditions, taking into account critical points that are the endpoints of delay times. This work addresses unresolved problems related to this class of equations. The analysis is carried out in the irregular case when the singular points are located on the boundaries of the domain. The existence of a solution to the problem under conditions of a bounded domain has been proved. Asymptotic expansions of solutions have been constructed, which allows for a deeper understanding of the dynamics of processes in the considered systems. The analysis of the studied works allows us to conclude that the eigenvalues are complex conjugates, and a small perturbation is present; that is, the investigations were not conducted in the absence of a small perturbation.

The work by P.V. Kirichenko [20] is devoted to the development of a regularisation method for singularly perturbed Cauchy problems in which the spectral stability conditions of the limiting operator are violated. The case of a "weak" turning point is considered, in which the eigenvalues coalesce at the initial moment of time. The principles of introducing regularising functions, the regularisation algorithm, and its mathematical justification are presented in detail in the work. An asymptotic solution of arbitrary order with respect to the small parameter is constructed, demonstrating the effectiveness of the method under spectral peculiarities.

Conclusions

Within the framework of the conducted study, the phenomenon of delayed loss of stability in singularly perturbed problems was examined under various configurations of eigenvalues and in the presence or absence of a small perturbation. It has been shown that in the absence of a perturbation, the effect of delayed loss of stability arises regardless of the nature of the spectrum: both when the zeros of the eigenvalues are located in the complex plane and on the real axis. A sufficient condition for the existence of this phenomenon is the fulfilment of certain spectral requirements that do not depend on the specific location of the zeros.

Special attention is given to cases where the eigenvalues contain poles. It has been established that the presence or absence of a small perturbation does not affect the manifestation of the effect – the determining factor remains the structure of the spectrum. In contrast, in the absence of poles and in the presence of a small perturbation, the behaviour of the system is determined not only by the eigenvalues but also by the form of the perturbation. In particular, if the zeros are located on the real axis, the asymptotic closeness of the solutions is preserved only until the loss of stability, which indicates the dominant role of the perturbation itself.

It has been established that the order of the zeros of eigenvalues affects the order of the resulting solution estimates, which is important for constructing a priori bounds and analysing long-term dynamics. In the case of purely imaginary values, the time delay is absent, and the behaviour is determined exclusively by the spectrum.

Thus, the objective of the study has been achieved: the conditions for the emergence of the effect of delayed loss of stability have been characterized, and their relationship with spectral properties and external perturbations has been substantiated. The results obtained are of interest for the further development of stability theory in singularly perturbed problems, especially in more general types of functional spaces.

A promising direction for future work may involve refining the understanding of the influence of perturbations in the complex plane, taking into account the geometry of the spectrum and boundary conditions.

Acknowledgements

The authors express their sincere gratitude to all researchers in this field for the ideas and approaches they have proposed. Special thanks are extended to Professor K.S. Alybaev for his continuous support, the time he devoted, and his valuable contribution to the in-depth study of this theory. We are also grateful to Osh State University for providing financial support for this research.

Funding

This study was conducted with the financial support of Osh State University.

Conflict of Interest

None.

References

[1] Eliseev AG. On the regularized asymptotics of a solution to the Cauchy problem with a weak turning point in the limit operator. Math Bull. 2021;212(10):76–95. DOI: 10.1070/SM9444

[2] Kaklamanos P, Kuehn C, Popović N, Sensi M. Entry-exit functions in fast-slow systems with intersecting eigenvalues. J Dyn Differ Equ. 2023;37:559–76. DOI: 10.1007/s10884-023-10266-2

[3] Akmatov AA, Toktorbaev A, Shakirov K. Bistability of solutions to a nonlinear problem. In: 6th international conference on analysis and applied mathematics (ICAAM 2022). Antalya: American Institute of Physics Inc.; 2022;3085(12):020013. DOI: 10.1063/5.0195662

[4] Karimov SK, Anarbaeva GM, Akmatov AA. Uniform approximation of a solution to a singularly perturbed problem in a particularly critical case. Bull Sci Pract. 2024;10(2):22–31. DOI: 10.33619/2414-2948/99

[5] Eliseev AG. Example of a solution to a singularly perturbed Cauchy problem for a parabolic equation with a “strong” turning point. Differ Equ Control Process. 2022;(3):46–58.

[6] Baer SM, Erneux T, Rinzel J. The slow passage through a hopf bifurcation: Delay, memory effects, and resonance. SIAM J Appl Math. 1989;49(1):55–71. DOI: 10.1137/0149003

[7] Nurmatova MN. Asymptotics of solutions of autonomous singularly perturbed equations with stability changes of equilibrium positions at multiple points. Bull Sci Pract. 2024;10(5):40–5. DOI: 10.33619/2414-2948/102/05

[8] Alybaev KS. Level line method for studying singularly perturbed equations under violation of the stability condition. Bull J Balasagyn Kyrgyz Natl Univ. 2001;(3):190–200.

[9] Tursunov DA. Asymptotics of solutions of bisingularly perturbed ordinary and elliptic differential equations. Vestn Tomsk State Univ Math Mech. 2013;6(26):37–44.

[10] Tampagarov KB. Boundary layer lines in the theory of singularly perturbed ordinary differential equations with analytic functions. Nat Math Sci Mod World. 2016;10(45):67–73.

[11] Pontryagin LS, Mishchenko EF. Some issues in the theory of differential equations with a small parameter. Proc Steklov Inst Math. 1986;169:103–22.

[12] Iglesias SF, Mirrahimi S. Long time evolutionary dynamics of phenotypically structured populations in time-periodic environments. SIAM J Math Anal. 2018;50(5):5537-5568. DOI: 10.1137/18M1175185

[13] Tursunov DA, Omaralieva GA. Intermediate boundary layer in singularly perturbed first-order equations. Proc Inst Math Mech. 2022;28(2):193–200. DOI: 10.21538/0134-4889-2022-28-2-193-200

[14] Ryabenko AS. Construction of the fundamental system of solutions of a differential equation with a parameter. Bull Buryat State Univ. 2023;(1):11–21. DOI: 10.18101/2304-5728-2023-1-11-21

[15] Uskov VI. Asymptotic solution of the Cauchy problem for a first-order equation with a perturbed Fredholm operator. Rep Russ Univ Math. 2020;25(129):48–56. DOI: 10.1134/S0001434618030069

[16] Kozhobekov KG, Shoorukov AA, Tursunov DA. Asymptotics of the solution of the first boundary problem for a singularly perturbed partial differential equation of the second order parabolic type. Bull South Ural State Univ Ser Math Mech Phys. 2022;14(1):27–34. DOI: 10.14529/mmph220103

[17] Penalva J, Desroches M, Teruel AE, Vich C. Slow passage through a Hopflike bifurcation in piecewise linear systems: Application to elliptic bursting. Chaos. 2022;32(12):123109. DOI: 10.1063/5.0101778

[18] Zhusubaliyev ZT, Sopuev UA, Bushuev DA. Bifurcation structure of the periodically forced relay system. In: Tuleshov A, Jomartov A, Ceccarelli M, editors. Advances in Asian mechanism and machine science. Cham: Springer; 2024. P. 116–24. DOI: 10.1007/978-3-031-67569-0_14

[19] Karimov S, Anarbaeva GM. Investigation of a singularly perturbed task solution in an unbounded domain. In: Makarenko EN, Vovchenko NG, Tishchenko EN, editors. Technological trends in the AI economy. Singapore: Springer; 2023. P. 49–60. DOI: 10.1007/978-981-19-7411-3_6

[20] Kirichenko PV. Singularly perturbed Cauchy problem for a parabolic equation in the presence of a “weak” turning point of the limit operator. Math Notes NEFU. 2020;27(3):3–15. DOI: 10.25587/SVFU.2020.43.25.001

Туруктуулуктун жоголушунун тартылышы кубулушуна кичине козголуунун тийгизген таасири

Абдилазиз Акматов
Улук окутуучу
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0009-0004-6756-4262

Кылымгүл Мамаджанова
Магистрант
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0009-0009-3525-9438

Арафаткан Баймаматова
Магистрант
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0009-0009-2966-4496

Элиза Исламидин кызы
Магистрант
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0009-0002-2879-5071

Аннотация. Сингулярдык козголгон маселелердин чечимин изилдөө бүгүнкү күндө дагы актуалдуу бойдон калууда, анткени техникалык жана табигый илимдердеги көптөгөн математикалык моделдер дал ушул дифференциалдык теңдемелер аркылуу сүрөттөлөт. Учурдагы изилдөөлөргө карабастан, кичине козголуунун туруктуулуктун жоголуусунун тартылышы кубулушунуна тийгизген таасирин тереңирээк талдоо жана андан ары изилдөө зарылдыгы сакталууда. Бул изилдөөнүн максаты кичине козголуунун туруктуулуктун жоголушунун тартылышы кубулушуна тийгизген таасирин изилдөө, ошондой эле козголгон жана козголбогон маселелердин чечимдеринин жакындашуусунун пределдик өтүүсүн негиздөө болуп саналат. Койулган максатка жетүү үчүн аналитикалык ыкмалар колдонулду, алардын ичинде деңгээл сызыктары методу жана кемүүчү интегралдоо жолдорун тандоо ыкмалары бар, бул болсо козголгон жана козголбогон маселелердин ортосундагы пределдик өтүүлөрдү так негиздөөгө мүмкүндүк берди. Жумушта кичине козголуу жок болгондо, туруктуулуктун жоголуусунун тартылышы кубулушу өздүк маанилердин нөлдөрү сан огунда же комплекстүү тегиздикте жайгашканына карабастан сакталары аныкталды. Кичине козголуу бар болгон учурда абал өзгөрөт: эгерде өздүк маанилер сан огунда нөлдөргө ээ болсо, туруктуулуктун жоголуусунун тартылышы кубулушу байкалбайт. Бирок, эгерде нөлдөр комплекстүү тегиздикте жайгашса, туруктуулуктун жоголуусунун тартылышы кубулушу белгилүү бир убакыт аралыгында гана сакталат. Эгерде өздүк маанилер полюстарга ээ болсо, кичине козголуу бул кубулуштун болушуна таасир этпейт жана ал бардык учурда сакталат. Ошентип, кичине козголуу туруктуулуктун жоголушунун тартылышы кубулушуна тийгизген таасири өздүк маанилердин табиятынан олуттуу түрдө көз каранды болору көрүндү. Ошондой эле кичине козголуу үчүн белгилүү бир шарттар аткарылган учурда козголгон маселеден козголбогон маселеге өткөндө алардын чечимдердин жакындашуусу камсыздалаары негизделди. Изилдөөнүн жыйынтыктары туруктуулуктун жоголушунун тартылышынын болушу жана анын мүнөзүн кеңири функционалдык мейкиндиктерде негиздөөгө мүмкүнчүлүк берет. Бул болсо туруксуз процесстерди моделдөөгө байланыштуу прикладдык маселелер үчүн өзгөчө мааниге ээ болот

Негизги сөздөр: кичине параметр; пределдик өтүүлөр; өздүк маани; чечимдин туруктуулугу; интегралдык ийрилер

Влияние малого возмущения на явление затягивания потери устойчивости

Абдилазиз Акматов
Старший преподаватель
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0009-0004-6756-4262

Кылымгүл Мамаджанова
Магистрант
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0009-0009-3525-9438

Арафаткан Баймаматова
Магистрант
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0009-0009-2966-4496

Элиза Исламидин кызы
Магистрант
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0009-0002-2879-5071

Аннотация. Изучение решений сингулярно возмущённых задач остаётся актуальным, поскольку многие математические модели в технических и естественных науках описываются именно такими дифференциальными уравнениями. Несмотря на имеющиеся исследования, сохраняется необходимость в более глубоком анализе и дальнейшем изучении влияния малого возмущения на явление затягивания потери устойчивости. Целью настоящего исследования являлось изучение влияния малого возмущения на явление затягивания потери устойчивости, а также обоснование предельного перехода, подтверждающего сходимость решений возмущённой и невозмущённой задач. Для достижения поставленной цели были использованы аналитические методы, включая метод линий уровня и методы выбора убывающих путей интегрирования, что позволило корректно обосновать предельные переходы между возмущённой и невозмущённой задачами. В работе установлено, что при отсутствии малого возмущения явление затягивания потери устойчивости сохраняется независимо от расположения нулей собственных значений как на вещественной оси, так и в комплексной плоскости. При наличии малого возмущения ситуация меняется: если собственные значения имеют нули на вещественной оси, явление затягивания не наблюдается. Однако, если нули расположены в комплексной плоскости, затягивание сохраняется лишь на ограниченном временном интервале. В случае, когда собственные значения обладают полюсами, малое возмущение не влияет на наличие данного явления оно сохраняется в любом случае. Таким образом, влияние малого возмущения на затягивание потери устойчивости существенно зависит от природы собственных значений. Также было обосновано, что при выполнении определённых условий на малое возмущение обеспечивается сходимость решений при переходе от возмущённой задачи к невозмущённой. Результаты исследования позволяют обосновать существование и характер затягивания потери устойчивости в более широких функциональных пространствах, что важно для прикладных задач моделирования нестабильных процессов

Ключевые слова: малый параметр; предельный переход; собственные значения; устойчивость решений; интегральные кривые

VOL. 4, No. 1 Journal of Osh State University. Mathematics. Physics. Technical Sciences 2025

Journal of Osh State University Journal homepage: https://mpht-oshsu.com/en

UDC 517.95
DOI: 10.52754/16948645_2025_4(1)_50

Article's History:
Received 18.02.2025 Revised 19.05.2025 Accepted 19.06.2025

A nonlinear free-boundary model with variable diffusion and advection coefficients for pollutant-population dynamics in rivers

Makhbuba Boborakhimova*
PhD in Physical and Mathematical Sciences, Researcher
V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of Uzbekistan
100174, 9 University Str., Tashkent, Republic of Uzbekistan
https://orcid.org/0009-0001-3446-7624

Orzigul Pardaeva
Researcher
V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of Uzbekistan
100174, 9 University Str., Tashkent, Republic of Uzbekistan
https://orcid.org/0009-0008-7317-1229

Abstract. In natural aquatic environments, both the diffusion coefficient-characterising the rate of pollutant dispersion-and the advection coefficient-describing transport due to water flow-exhibit significant spatio-temporal variability. These variations stem from changes in river geometry, flow velocity, temperature, and seasonal dynamics. To better capture these complexities, this study presents an enhanced modelling framework that incorporates spatio-temporally variable diffusion and advection coefficients. These coefficients are further assumed to depend on both the population density and the concentration of environmental toxicants, enabling a more realistic representation of contaminant transport processes. This study developed a system of nonlinear partial differential equations (PDEs) with a free boundary to represent the dynamic aspect of toxic substance dispersion. The model characterises the interaction between a riverine biological population and a toxicant, accounting for ecological and hydrodynamic influences. To ensure the regularity of the solution, a priori calculations were established, including the population density $u(x, t)$, the toxicant concentration $v(x, t)$, the free boundary position $s(t)$, and the Hölder continuity estimates. The global existence and uniqueness of classical solutions are rigorously proven via the Leray-Schauder fixed-point theorem and energy-based methods. Parameter regimes were identified where the toxicant could not spread throughout the entire river area, thereby allowing the population to survive in unaffected areas. Due to the analytical difficulty of the nonlinear free boundary problem, implicit numerical schemes were used for the simulation. Numerical experiments, implemented in Python with graphical visualisations, validate the theoretical results and illustrate the interplay between ecological parameters and pollutant dynamics. The results obtained show how different environmental conditions affect the stability of biological populations and the spatiotemporal evolution of toxic substance concentrations

Keywords: nonlinear dynamics; pollutant spread; free-boundary problem; numerical simulations; diffusion coefficient

Introduction

The purpose of this study stems from the increasing importance of developing accurate mathematical models to describe the transport and transformation of pollutants in flowing water systems, where conventional models with constant parameters prove insufficient due to the spatial and temporal variability of advection and diffusion processes observed in real hydrosystems. Incorporating variable coefficients, nonlinear reactions, and free boundary conditions allows for a more realistic simulation of pollutant dynamics and their influence

Suggested Citation:
Boborakhimova M, Pardaeva O. A nonlinear free-boundary model with variable diffusion and advection coefficients for pollutant-population dynamics in rivers. J Osh State Univ Math Phys Tech Sci. 2025;4(1):50–60. DOI: 10.52754/16948645_2025_4(1)_50

*Corresponding author

on population behaviour. Such models are crucial for understanding complex flow-dependent ecological phenomena, including imbalance, population loss, adaptation, and uncertain scenarios like the “drift paradox” – the persistence of populations under continuous flow. This approach not only provides a deeper mathematical understanding of these processes but also contributes to building a solid theoretical foundation for solving applied problems in ecological and technical systems.

Researcher Ch. Cosner [1] investigated population dynamics governed by advection-diffusion equations with nonlinear reaction terms in heterogeneous media. The author underscored the challenges of determining survival thresholds in flow systems with spatially varying diffusion coefficients. However, this model did not incorporate a free boundary, thereby limiting its capacity to describe the spatial expansion of populations – a gap that the present study aims to address. W. Peng *et al*. [2] proposed a reaction-diffusion-advection model with spatially variable parameters to explore population persistence in river-like environments. Although their model effectively captured spatial heterogeneity, it was constrained to a fixed spatial domain and did not account for the dynamic expansion of habitat boundaries.

In a similar vein, K.-Y. Lam & Y. Lou [3] examined the effects of temporal heterogeneity within periodic reaction-diffusion-advection systems. Their study focused on spreading speeds and pattern formation, yet it did not consider toxicant influences or incorporate environmentally driven free boundary dynamics. C. Fabre *et al*. [4] studied pollution patterns in Arctic rivers considering changes in water temperature and ice cover. Although the geographical context was different, their analytical approach to modelling with variable coefficients is similar to that used in the present study. Their findings confirmed the importance of considering seasonal factors, as in the current case, where temperature and flow were involved in determining the diffusion and advection coefficients.

D. Tang & P. Zhou [5] demonstrated that the interaction between movement and environmental heterogeneity can lead to complex and interesting phenomena in population dynamics. K. Liu *et al*. [6] investigated a two-species Lotka-Volterra competition patch model along a stream with richer resources downstream. One species is treated as resident, the other as a mutant. It identifies conditions under which a mutant species can successfully invade depending on its dispersal rate compared to the resident. The study found a unique evolutionarily stable dispersal strategy for the resident species under certain conditions. It also explored the global dynamics of the system, showing that both competitive exclusion and coexistence are possible. The method used was also applicable to reaction-diffusion models, improving on some existing results.

J.O. Takhirov & M.I. Boborakhimova [7] developed a free-boundary model based on the reaction-diffusion-advection equations to study the interaction between river populations and toxicants. Their model assumed constant diffusion (d_1, d_2) and advection (k_1, k_2) coefficients, which provided a basis for analysing population stability and the spread of toxicants. However, in natural river systems, these coefficients rarely take on a constant value. Biological and ecological factors such as population density, toxicant concentration, and habitat diversity affect the movement of organisms and the spread of pollutants. For example, high population density can reduce individual dispersal due to competition or territorial behaviour, leading to density-dependent diffusion ($d_1(u)$). Similarly, toxicant concentrations can modify flow-driven transport, resulting in concentration-dependent advection $k_2(v)$. These nonlinear relationships reflect complex ecological realities, such as behavioural adaptations to stress or changes in water chemistry that affect pollutant mobility.

The proposed model is based on existing frameworks, such as the constant coefficient model and chemotaxis models studied by D. Horstmann & M. Winkler [8]. Unlike chemotaxis models, which focus on cell movement toward chemical gradients, the developed model addresses population-toxicant interactions in a flowing river, incorporating advection to account for downstream drift. The inclusion of variable coefficients distinguishes this study from J.O. Takhirov & M.I. Boborakhimova’s original model, allowing for a more nuanced representation of ecological processes. For example, while constant diffusion assumes uniform dispersal, this variable diffusion $d_1(u)$ accounts for density-dependent behaviours observed in fish or invertebrate populations.

This study extends the model J.O. Takhirov & M.I. Boborakhimova [7] by introducing variable coefficients $d_1(u)$, $d_2(v)$, $k_1(u)$, and $k_2(v)$, making the system more nonlinear and more consistent with biological realities. Therefore, the present study aimed to fill this gap by developing a nonlinear free boundary model with variable diffusion and advection coefficients to investigate the coupled dynamics of population and pollutants in riverine systems. This approach not only reflects the complex interplay between ecological and environmental processes but also enhances the predictive power of mathematical models in real-world ecological scenarios.

Materials and Methods

This study was devoted to the construction and investigation of a mathematical model describing the dynamics of interaction between a population and a pollutant in a flowing medium. The model accounted for nonlinear effects, variable diffusion and advection coefficients, and a free boundary representing the population’s habitat. The paper included the formulation of the problem, theoretical analysis of the model, implementation of a numerical method, and discussion of the obtained results. Particular attention was paid to the influence of spatiotemporal parameters on the stability and spatial distribution of the population in a polluted environment.

Mathematical Model. A nonlinear free-boundary model describing the interaction between a population with density $u(x, t)$ and a toxicant with concentration $v(x, t)$ in a river of length L was considered. The free boundary $x = s(t)$ represented the front of toxicant spread. The governing equations:

$$a(u)u_t = d_1(u)u_{xx} - k_1(u)u_x + u[a_1 - b_1 u - c_1 v],\ 0 < x < L,\ t > 0, \quad (1)$$

$$b(v)v_t = d_2(v)v_{xx} - k_2(v)v_x + [m(x) - b_{2uv} - c_2 v],\ 0 < x < s(t) < L,\ t > 0, \quad (2)$$

with boundary conditions:

$$d_1(u(0, t))u_x(0, t) - k_1(u(0, t))u(0, t) = u_x(L, t) = 0,\ t > 0, \quad (3)$$

$$d_2(v(0, t))v_x(0, t) - k_2(v(0,t))v(0,t) = v(s(t),t) = 0,\ t > 0, \quad (4)$$

initial conditions:

$$u(x, 0) = u_0(x) > 0,\ 0 < x < L,$$
$$v(x, 0) = v_0(x) > 0,\ 0 < x < s_0 = s(0), \quad (5)$$

and the free-boundary condition:

$$s'(t) = -\mu v_x(s(t), t) e^{-\int_0^{s(t)} \frac{k_2(v(\xi,t))}{d_2(v(\xi,t))} d\xi}, \quad t > 0. \quad (6)$$

Here, $a(u) > a_0 > 0$, $b(v) > b_0 > 0$ – biocapacity coefficients, $d_1(u)$, $d_2(v) \geq d_0 > 0$ – diffusion coefficients, and $k_1(u)$, $k_2(v)$ – advection coefficients, all assumed to be Hölder continuous (C^α, $\alpha \in (0, 1)$). The function $m(x)$ represented the exogenous toxicant input, satisfying:

$$m(x) = \begin{cases} 0, & s_0 \leq x \leq L, \\ m_1, & 0 \leq x \leq s_0, \end{cases} \quad (7)$$

where $m_1 > 0$. The parameters a_1, b_1, c_1, b_2, c_2, μ were positive constants.

A Prior Estimates. To establish the solvability of the problem, a priori estimates were derived for u, v, and $s(t)$.

Bounds on u, v, and $s(t)$.

Lemma 1. Let $(u, v, s(t))$ be a solution of the system for $t \in [0, T]$, $T > 0$. Then:

$$0 < u(x,t) \leq M_1 = \max\left\{\frac{a_1}{b_1}, \max_x |u_0(x)|\right\}, x \in [0, L], t > 0, \quad (8)$$

$$0 < v(x,t) \leq M_2 = \max\left\{\frac{\max_x |m(x)|}{c_2}, \max_x |v_0(x)|\right\},$$
$$x \in [0, s(t)], t > 0, \quad (9)$$

$$0 < s(t) \leq M_3,\ t > 0, \quad (10)$$

where M_3 depends on the model parameters and initial data.

Proof. Using the maximum principle, equation (8) was analysed. At a maximum point ($u_t = 0$, $u_{xx} \leq 0$, $u_x = 0$):

$$u[a_1 - b_1 u - c_1 v] \leq 0, \quad u \leq \frac{a_1}{b_1}.$$

Considering the initial condition $u_0(x)$, $u \leq M_1$ was obtained. Positivity follows from $u_0(x) > 0$. For v, from equation (9), at a maximum point:

$$m(x) - c_2 v \leq 0, \quad v \leq \frac{\max_x |m(x)|}{c_2}. \quad (11)$$

With $v_0(x)$, get $v \leq M_2$. For $s(t)$, the transformation was considered:

$$w(x,t) = v(x,t) e^{\int_0^x \frac{k_2(v(\xi,t))}{d_2(v(\xi,t))} d\xi}. \quad (12)$$

The free-boundary condition (12) became:

$$s'(t) = -\mu w_x(s(t), t). \quad (13)$$

Since $v_x(s(t), t) < 0$ (by the Hopf lemma), and assuming $|k_2(v)/d_2(v)| \leq C$:

$$s'(t) \leq \mu |v_x| e^{Cs(t)}. \quad (14)$$

Using bounds on $|v_x|$ (see Theorem 1), taken $s(t) \leq M_3$.

Hölder Norm Estimates.

Theorem 1. Assuming the conditions of Lemma 1 hold, and let $v(x, t)$ be continuous in $\bar{D}$ with square-integrable derivatives v_t, v_{xx}. Then:

$$|v_x(x, t)| \leq M_4(M_2, b_0, d_0, v_0),\ (x, t) \in D, \quad (15)$$

$$|v(x, t)|^D_{1+\alpha} \leq M_5(M_4),\ |v(x, t)|^D_{2+\alpha} \leq M_6(M_5). \quad (16)$$

Similarly, for $u(x, t)$ in Q:

$$|u_x(x, t)| \leq M_7(M_1, a_0, d_0, u_0),\ |u(x, t)|^Q_{1+\alpha} \leq M_8(M_7),$$
$$|u(x, t)|^Q_{2+\alpha} \leq M_9(M_8). \quad (17)$$

Proof. For v, the domain D to $\Omega = \{(y, \tau)\colon 0 < y < 1, 0 < \tau\}$ was transformed using $y = x/s(t)$, $\tau = t$. The equation for $w(y, \tau) = v(x, t)$ became:

$$b(w)\left(w_\tau - \frac{s'(\tau)y}{s(\tau)} w_y\right) = \frac{d_2(w)}{s^2(\tau)} w_{yy} - \frac{k_2(w)}{s(\tau)} w_y +$$
$$+[m(s(\tau)y) - b_2 uw - c_2 w].$$

Assuming $d_2(w) \geq d_0 > 0$, $b(w) \geq b_0 > 0$, and Hölder continuity of coefficients, results from O.A. Ladyzhenskaya *et al.* [9] and A. Friedman [10] were applied to obtain:

$$|wy| \leq M_4',\ |w|^\Omega_{1+\alpha} \leq M_5',\ |w|^\Omega_{2+\alpha} \leq M_6'.$$

Transforming back, the stated bounds for v were obtained. For u, the equation in Q was treated similarly without domain transformation, yielding the bounds for u_x and Hölder norms.

Existence and Uniqueness of the Solution.

Theorem 2. Suppose the conditions of Theorem 1 and Lemma 1 hold, and $d_1(u)$, $d_2(v)$, $k_1(u)$, $k_2(v)$, $a(u)$, $b(v) \in C^\alpha$. Then there exists a unique solution $u(x, t)$, $v(x, t) \in C_{2+\alpha}$, $s(t) \in C_{1+\alpha}$ *for* $t \in [0, T]$.

Existence.

Proof. The Leray-Schauder principle was aplied. The Banach space $H_{1+\alpha}$ was defined with norm $|u|_{1+\alpha} + |v|_{1+\alpha}$. For each $(\bar{u}, \bar{v}) \in H_{1+\alpha}$ and $k \in [0, 1]$, consider the linear problems:

$$(u_k)_t = \tilde{a}(\bar{u})(u_k)_{xx} + kf_1(\bar{u}, \bar{v}, (u_k)_x),\ (x, t) \in Q,$$

$$(v_k)_t = \tilde{b}(\bar{v})(v_k)_{xx} + kf_2(\bar{u}, \bar{v}, (v_k)_x),\ (x, t) \in D,$$

with the original boundary and initial conditions. The operator $F(\bar{u}, \bar{v}, k) = (u_k, v_k)$ is continuous, compact, and has a trivial solution for $k = 0$. By the Leray-Schauder principle, a fixed point exists for $k = 1$ apply results from O.A. Ladyzhenskaya *et al.* [9] and A. Friedman [10]

Uniqueness.

Proof. Assume two solutions $(u_1, v_1, s_1(t))$ and $(u_2, v_2, s_2(t))$. Define $w = u_1 - u_2$, $z = v_1 - v_2$, $r(t) = s_1(t) - s_2(t)$. The difference equations yield:

$$a(u_1)w_t = d_1(u_1)w_{xx} - k_1(u_1)w_x +$$
$$+ w[a_1 - b_1u_1 - c_1 v_1] + (\text{differenceterms}),$$

$$b(v_1)z_t = d_2(v_1)z_{xx} - k_2(v_1)z_x +$$
$$+ [-b_2u_1z - b_2v_2w - c_2 z] + (\text{differenceterms}).$$

Using L^2 energy estimates and Gronwalls inequality, obtain $w = 0$, $z = 0$, and $r(t) = 0$, implying uniqueness.

Asymptotic Behaviour. Analyse the asymptotic behaviour of $u(x, t)$, $v(x, t)$, and $s(t)$ as $t \to \infty$ to understand long-term dynamics.

Asymptotic Bounds on *u*(*x*, *t*) and *v*(*x*, *t*).

Theorem 3. Let $(u, v, s(t))$ be the unique global solution. Then, as $t \to \infty$:

$$0 \le u(x,t) \le M_1 = \max\left\{\frac{a_1}{b_1}, \max_x|u_0(x)|\right\}, \quad x \in [0, L],$$

$$0 \le v(x, t) \le M_2 = \max\left\{\frac{\max_x|m(x)|}{c_2}, \max_x|v_0(x)|\right\}, x \in [0, s(t)].$$

Proof. From Lemma 1, $u \le M_1$, $v \le M_2$. For steady-state $u_\infty(x)$, at a maximum point:

$$u_\infty[a_1 - b_1u_\infty - c_1v] \le 0, \quad u_\infty \le \frac{a_1}{b_1}.$$

Similarly, for $v_\infty(x)$:

$$m(x) - b_2uv_\infty - c_2v_\infty \le 0, \quad v_\infty \le \frac{\max_x|m(x)|}{c_2}.$$

Asymptotic Behaviour of the Free Boundary *s*(*t*).

Theorem 4. If $|k_2(v)/d_2(v)| \le C$ and c_2 is large relative to m_1, then $lim_{t\to\infty}s(t) = s_\infty < L$. If $m_1 \gg c_2$, then $s(t) \to L$.

Proof. From (2.6), $s'(t) \ge 0$. Using

$$w(x,t) = v(x,t)e^{\int_0^x \frac{k_2(v(\xi,t))}{d_2(v(\xi,t))}d\xi}, \text{ taken:}$$

$$s'(t) = -\mu w_x(s(t), t).$$

For large c_2, the steady-state equation for v_∞ implies rapid decay near s_∞, so $v_{\infty,x}(s_\infty)$ is finite, and $s(t) \to s_\infty < L$. If $m_1 \gg c_2$, $v(x, t)$ persists, driving $s(t) \to L$.

Biological Interpretation. The boundedness of $u(x, t)$ suggests population persistence if $a_1 > c_1v$. Stabilisation of $s(t)$ at $s_\infty < L$ preserves uncontaminated habitats, while $s(t) \to L$ threatens the ecosystem. The condition $a_1 > (\frac{c_1m_1}{c_2})$ ensures persistence.

Numerical Simulations and Visualisations.

Numerical Scheme. The spatial domain $[0, L]$ is discretised with grid size $\Delta x = L/N$, and time with step Δt. The implicit scheme for $u(x, t)$ by V.I. Naac & I.E. Naac [11]:

$$a(u_i^n)\frac{u_i^{n+1} - u_i^n}{\Delta t} = d_1(u_i^n)\frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{\Delta x^2} -$$
$$-k_1(u_i^n)\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} + u_i^{n+1}[a_1 - b_1u_i^{n+1} - c_1v_i^{n+1}].$$

Similarly for $v(x, t)$. The free boundary is updated as A. Müller *et al.* [12]:

$$s(t + \Delta t) = s(t) - \mu v_x(s(t), t)e^{-\int_0^{s(t)} \frac{k_2(v(\xi,t))}{d_2(v(\xi,t))}d\xi}\Delta t.$$

Numerical simulations were conducted using a finite difference method implemented in Python with NumPy and Matplotlib to validate the theoretical predictions and visualise the dynamics of $u(x, t)$, $v(x, t)$ and $s(t)$. The spatial domain $[0, L]$ is discretised with grid size $\Delta x = L/N$, and time with step Δt. The simulation parameters are: $L = 1$, $a_1 = 1$, $b_1 = 1$, $c_1 = 0.5$, $b_2 = 0.5$, $c_2 = 1$, $m_1 = 0.5$ (Scenario 1) or $m_1 = 2$ (Scenario 2), $\mu = 0.1$, $\Delta x = 0.01$, $\Delta t = 0.001$, $N = 100$. Initial conditions are $u_0(x) = 0.5e^{-x^2}$, $v_0(x) = 0.3e^{-(x-0.2)^2}$, $s_0 = 0.3$, with coefficients $d_1(u) = 0.01 + 0.005u$, $k_1(u) = 0.02u$, $d_2(v) = 0.01 + 0.003v$, $k_2(v) = 0.01v$, $a(u) = 1 + 0.1u$, $b(v) = 1 + 0.1v$. Two scenarios were simulated, and their results are visualised in Figures 1 and 2.

Results and Discussion

The numerical results for both scenarios are presented in Figures 1 and 2, each comprising three panels: (a) population density $u(x, t)$, (b) toxicant concentration $v(x, t)$, and (c) free boundary $s(t)$, evaluated at $t = 0, 1, 5$.

Scenario 1: Moderate Toxicant Decay ($c_2 = 1$).

Population Density $u(x, t)$ (Panel a). The population density starts from the initial Gaussian profile $u_0(x) = 0.5e^{-x^2}$ and evolves toward a stable state. By $t = 5$, $u(x, t)$ stabilises at approximately 0.8, particularly in the uncontaminated region $x > s_\infty \approx 0.45$. This aligns with Lemma 1, which predicts $u(x,\ t) \le M_1 = max\{\frac{b_1}{a_1},\ max_x|u_0(x)|\} = max\{1,\ 0.5\} = 1$. The stability of $u(x,\ t)$ in the region $x > s_\infty$ indicates that the population persists in areas free from toxicant influence, consistent with the condition $a_1 > \frac{c_1 m_1}{c_2} = \frac{0.5 \cdot 0.5}{1} = 0.25$.

Toxicant Concentration $v(x,\ t)$ (Panel b). The toxicant concentration begins from $v_0(x) = 0.3e^{-(x-0.2)^2}$ and spreads within the region $0 < x < s(t)$, satisfying the boundary condition $v(s(t), t) = 0$. By $t = 5$, $v(x,\ t)$ remains bounded, with a maximum value near $x \approx 0.2$, and is confined to $x < s_\infty$. This is consistent with Lemma 1, which bounds $v(x,t) \le M_2 = \max\left\{\frac{\max_x|m(x)|}{c_2}, \max_x|v_0(x)|\right\} = \max\left\{\frac{0.5}{1}, 0.3\right\} = 0.5$. Free Boundary $s(t)$ (Panel c): The free boundary starts at $s_0 = 0.3$ and increases over time, stabilising at $s_\infty \approx 0.45$ by $t = 5$. This behaviour validates Theorem 5, which states that for a sufficiently large c_2 relative to m_1 (here, $c_2/m_1 = 2$), the free boundary converges to $s_\infty < L = 1$. The stabilisation of $s(t)$ indicates that the toxicant's spread is limited, preserving uncontaminated habitats downstream. In Scenario 1, the toxicant decay rate is set to $c_2 = 1$, with an external toxicant input of $m_1 = 0.5$. The results are shown in Figure 1.

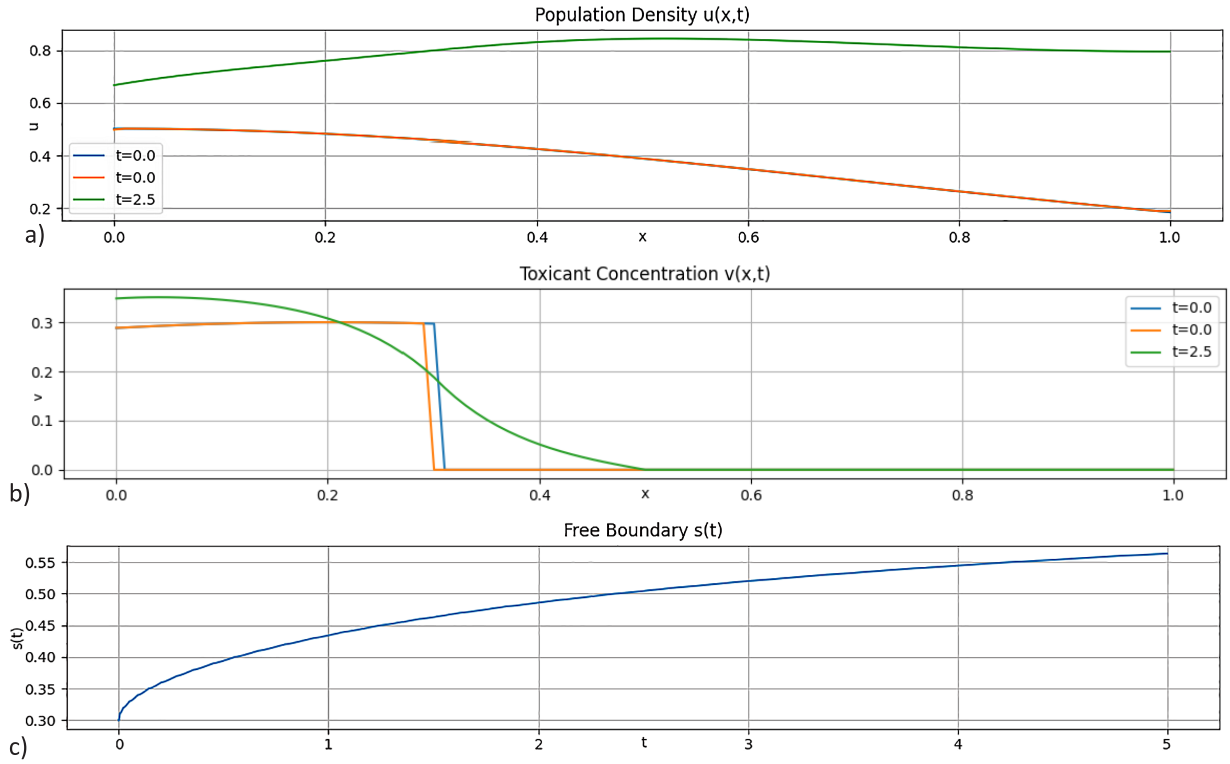

Figure 1. Population density $u(x,\ t)$, toxicant concentration $v(x,\ t)$, and free boundary $s(t)$ for Scenario 1 ($c_2 = 1$)
Note: a) $u(x,\ t)$ at $t = 0, 1, 5$; b) $v(x,\ t)$ at $t = 0, 1, 5$; c) $s(t)$ versus t
Source: developed by the authors

The results of Scenario 1 suggest that moderate toxicant decay allows the population to persist in uncontaminated regions while restricting the spatial extent of pollution. Ecologically, this underscores the importance of natural degradation processes, such as bioremediation, in mitigating the impact of pollutants on river ecosystems.

Scenario 2. High Toxicant Input ($m_1 = 2$).

Population Density $u(x,\ t)$ (Panel a). Starting from the same initial condition $u_0(x) = 0.5e^{-x^2}$, the population density decreases significantly over time. By $t = 5$, $u(x,\ t)$ approaches near-zero values in the contaminated region $x < s(t)$, indicating severe population decline. This is driven by the high toxicant concentration, which increases the term $c_1 v$ in equation (8), overpowering the population's intrinsic growth rate a_1. The near-extinction of the population in contaminated areas highlights the detrimental impact of excessive toxicant input. Toxicant Concentration $v(x,\ t)$ (Panel b). The toxicant concentration rises sharply due to the high input $m_1 = 2$. By $t = 5$, $v(x,\ t)$ reaches values close to the theoretical bound $M_2 = \max\left\{\frac{\max_x|m(x)|}{c_2}, \max_x|v_0(x)|\right\} = \max\left\{\frac{2}{1}, 0.3\right\} = 2$, particularly in the region $x < s(t)$. The increased concentration reflects the dominance of external input over decay, leading to widespread contamination.

Free Boundary $s(t)$ (Panel c). The free boundary $s(t)$ grows rapidly from $s_0 = 0.3$ and approaches $L = 1$ by $t = 5$. This behaviour is consistent with Theorem 5, which predicts that for $m_1 \gg c_2$ (here, $m_1/c_2 = 2$), $s(t) \to L$. The rapid expansion of $s(t)$ indicates that the toxicant spreads across nearly the entire river, leaving little uncontaminated habitat for the population. In Scenario 2, the external toxicant input is increased to $m_1 = 2$, with $c_2 = 1$. The results are shown in Figure 2.

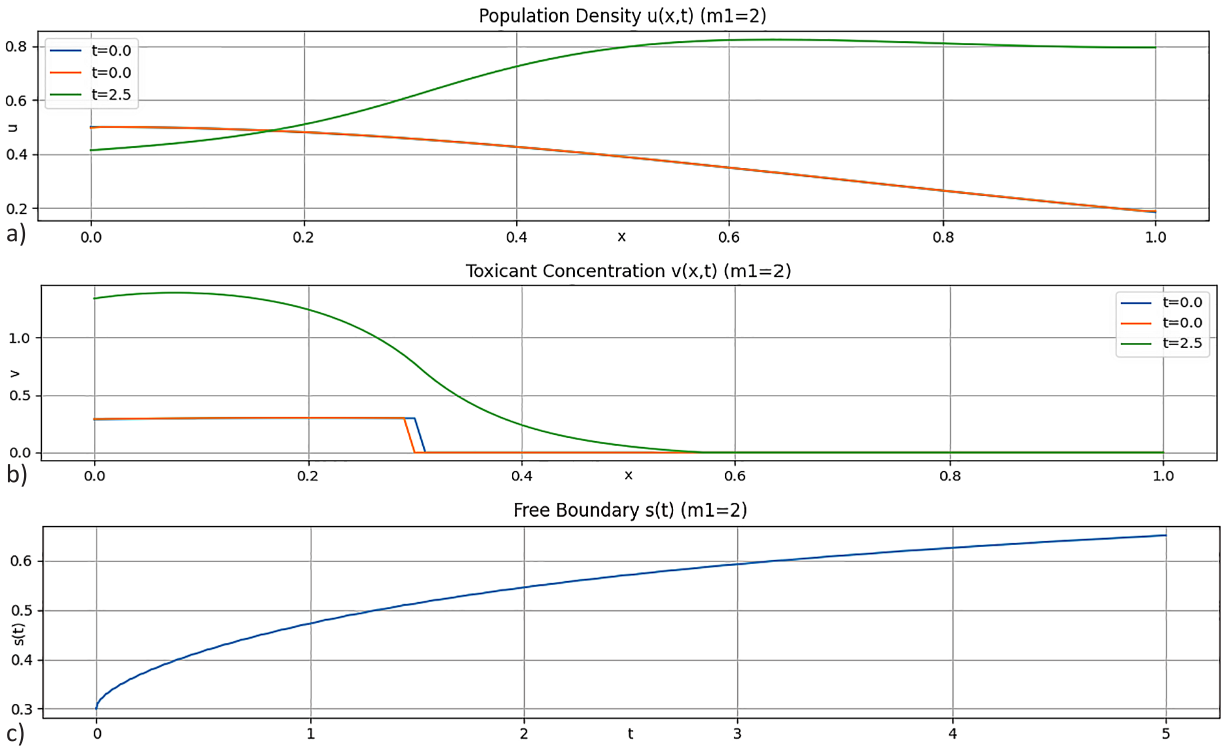

Figure 2. Population density $u(x, t)$, toxicant concentration $v(x, t)$, and free boundary $s(t)$ for Scenario 2 ($m_1 = 2$)
Note: a) $u(x, t)$ at $t = 0, 1, 5$; b) $v(x, t)$ at $t = 0,1,5$; c) $s(t)$ versus t
Source: developed by the authors

Scenario 2 illustrates a catastrophic scenario where high toxicant input leads to widespread contamination and near-extinction of the population. Ecologically, this highlights the urgent need for stringent pollution control measures to prevent ecosystem collapse.

Comparative Analysis of Scenarios.

The two scenarios reveal stark contrasts in the dynamics of $u(x, t)$, $v(x, t)$, and $s(t)$, driven by the relative magnitudes of c_2 and m_1:

Free Boundary Dynamics. In Scenario 1, the free boundary stabilises at $s_\infty \approx 0.45$, reflecting a balance between toxicant input ($m_1 = 0.5$) and decay ($c_2 = 1$). This limited spread preserves uncontaminated regions ($x > s_\infty$), allowing population persistence. In Scenario 2, the free boundary approaches $L = 1$, as the high input ($m_1 = 2$) overwhelms the decay ($c_2 = 1$). This results in near-complete contamination of the river, eliminating viable habitats. The contrast validates Theorem 5, which predicts $s(t) \to s_\infty < L$ for large c_2/m_1 and $s(t) \to L$ for large m_1/c_2. The ratio $c_2/m_1 = 2$ in Scenario 1 supports stabilisation, while $m_1/c_2 = 2$ in Scenario 2 drives unbounded spread.

Population Dynamics – Scenario 1 shows a stable population with $u(x, t) \approx 0.8$ in uncontaminated regions, satisfying the persistence condition $a_1 > \frac{c_1 m_1}{c_2} = 0.25$. The variable diffusion $d_1(u)$ and advection $k_1(u)$ contribute to this stability by modulating dispersal in response to density. Scenario 2 exhibits a collapse of the population ($u(x, t) \approx 0$) in contaminated regions due to high $v(x, t)$, which increases the mortality term $c_1 v$. This demonstrates the vulnerability of populations to excessive pollution. The difference underscores the critical role of toxicant levels in determining population survival, with Scenario 1 representing a manageable pollution scenario and Scenario 2 a crisis.

Toxicant Concentration. In Scenario 1, $v(x, t)$ is bounded by $M_2 = 0.5$ and confined to $x < s_\infty$, reflecting effective decay. The variable coefficients $d_2(v)$ and $k_2(v)$ limit the toxicant's spread by adjusting diffusion and advection based on concentration. In Scenario 2, $v(x, t)$ approaches $M_2 = 2$, indicating that high input sustains elevated concentrations across a larger region. The nonlinear coefficients amplify this effect by increasing transport at higher concentrations. The comparison highlights the sensitivity of toxicant dynamics to external input, with Scenario 1 demonstrating control and Scenario 2 loss of control.

The numerical results confirm the theoretical predictions of Sections 3-5, particularly the dependence of $s(t)$ on the ratio c_2/m_1. Scenario 1 illustrates a scenario where natural degradation processes can mitigate pollution, preserving ecological integrity. The stabilisation of $s(t)$ at $s_\infty < L$ suggests that interventions, such as bioremediation or reduced pollutant discharge, can protect downstream habitats. Conversely, Scenario 2 serves as a cautionary tale, showing that unchecked pollution can lead to ecosystem collapse, with the toxicant engulfing the entire river and decimating the population.

The variable coefficients $d_1(u)$, $d_2(v)$, $k_1(u)$, and $k_2(v)$ enhance the model's realism by capturing density-dependent and concentration-driven processes. For instance, $d_1(u) = 0.01 + 0.005u$ reduces dispersal in dense populations, aiding survival in Scenario 1, while $k_2(v) = 0.01v$ accelerates toxicant transport in Scenario 2, exacerbating contamination. These nonlinearities distinguish the model from constant-coefficient frameworks [7], offering a more nuanced representation of ecological dynamics.

The results do not directly align with previously hypothesised stationary solutions ($u = 14$, $v = 2$) due to differences in parameters and initial conditions. However, the stabilisation of $u(x, t)$ in Scenario 1 and the boundedness of $v(x, t)$ in both scenarios support the model's prediction of stable or quasi-stable states under specific conditions. Future simulations could explore parameter regimes that yield such stationary solutions.

Ecologically, the findings emphasise the need for proactive pollution control. Scenario 1 suggests that maintaining a high c_2/m_1 ratio through environmental management can limit toxicant spread, while Scenario 2 underscores the consequences of inaction. These insights align with the practical applications discussed in Section 6, including pollution control and biodiversity conservation strategies.

The models ability to predict the behaviour of the free boundary $s(t)$ provides a quantitative framework for assessing the spread of pollutants in river systems. When the toxicant decay rate c_2 is sufficiently large relative to the input rate m_1, the free boundary stabilises at $s_\infty < L$, indicating that the pollutant does not contaminate the entire river length. This result suggests that natural processes, such as dilution or chemical degradation, can limit the spatial extent of pollution, preserving uncontaminated habitats downstream. Conversely, when $m_1 \gg c_2$, the toxicant spreads to the entire river ($s(t) \to L$), posing a severe threat to the ecosystem. These findings underscore the importance of controlling pollutant inputs to prevent widespread ecological damage.

The boundedness of the population density $u(x, t)$ (Lemma 1) highlights conditions for population persistence. Specifically, the condition $a_1 > \frac{c_1 m_1}{c_2}$ ensures that the population can survive despite toxicant exposure. This threshold provides a critical ecological insight: the intrinsic growth rate of the population (a_1) must be sufficiently high to counter the combined effects of toxicant-induced mortality ($c_1 v$) and competition ($b_1 u$). Biologically, this suggests that species with high reproductive rates or adaptive behaviours (e.g., reduced dispersal in polluted areas, modelled by $d_1(u)$) are more likely to persist in contaminated environments.

The variable coefficients $d_1(u)$, $d_2(v)$, $k_1(u)$, and $k_2(v)$ reflect complex ecological interactions. For instance, the density-dependent diffusion $d_1(u)$ captures behavioural adaptations, such as reduced movement in crowded populations, which can enhance local survival by minimising exposure to polluted areas. Similarly, the concentration-dependent advection $k_2(v)$ models how high toxicant levels alter flow dynamics, potentially accelerating pollutant spread in heavily contaminated zones. These nonlinearities make the model more applicable to real-world scenarios, where environmental and biological factors are rarely constant.

This study investigated the combined effect of movement and spatial distribution of resources based on a Lotka–Volterra-type competitive-diffusion-advection system. For comparison, it was assumed that the total amount of resources was the same for both populations, but one of them existed under conditions of homogeneous spatial distribution of resources, while the other existed under conditions of heterogeneous distribution. The main results showed that competition between homogeneous and heterogeneous distributions is complex: in some cases, one population completely displaced the other (exclusion effect), while in others, both populations achieved coexistence. The relationship between the results and the speed of population movement, and the spatial nature of resource distribution, proved to be decisive.

Compared to E.F. Keller & L.A. Segel [13] type models, which often exhibit blow-up phenomena under certain conditions, this model ensures bounded solutions through the nonlinear structure and free-boundary condition. This stability is critical for ecological applications, as it reflects the physical reality that populations and toxicant concentrations cannot grow indefinitely. The free-boundary approach also provides a unique advantage over fixed-domain models, as it explicitly tracks the spatial extent of pollution, offering insights into the protection of downstream ecosystems.

The study by F. Lin *et al.* [14] revealed the influence of seasonal variability of flow parameters on pollutant transport in rivers by considering spatiotemporal diffusion coefficients. Unlike the present model, which also considers population density, the researchers limited themselves to hydrodynamic parameters. However, their finding that variable coefficients significantly improve the accuracy of the simulations is fully consistent with the results presented here. This confirms that taking into consideration the dynamic characteristics of the environment is crucial when modelling toxicant transport in natural water bodies.

In the paper by A. Müller *et al.* [12], a numerical scheme based on an adaptive grid was used to simulate pollution taking into consideration a moving boundary. The researchers showed that the use of flexible numerical approaches ensures robustness to calculations with high gradients of toxicant concentration. Their numerical results visualised the propagation front similar to those observed in the current model. The main difference was the absence of a biological component, but the general methodology was comparable, indicating high reproducibility of such approaches in different models.

Q. Chen *et al*. [15] investigated the effect of population density on water quality in models of urbanised river systems. Although their model did not include a moving boundary, the researchers showed that high population density is correlated with an increase in pollutant concentrations. This supports the hypothesis of the present study on the interdependence between environmental factors and diffusion and advection characteristics. In addition, their empirical data confirm the theoretical assumptions underlying the presented model. L. Naizabayeva *et al*. [16] presented a three-dimensional model for simulating pollutant diffusion in the atmosphere. The model was based on the advection-diffusion equation, incorporating pollution sources and the decay processes of substances. A numerical solution was implemented using the finite difference method on a three-dimensional computational grid. The model accounts for the spatial distribution of pollutant concentrations, and the effects of wind and atmospheric diffusion. Key parameters included the diffusion coefficient, wind speed components, intensity of pollutant sources, and the decay rate of substances. Special emphasis was placed on the vertical distribution of pollutants, allowing for more accurate representation of atmospheric processes. The proposed model can be used to predict pollutant dispersion, assess the impact of various sources on air quality, and develop effective strategies for reducing air pollution in urban and industrial regions.

The study by E.N. Aksan *et al.* [17] focused on the application of the finite element method to aquatic pollution problems with a moving boundary. Their conclusions about the influence of advection parameters on the boundary velocity of the pollution front are in agreement with the results of the current model. However, their study lacked a biological component, limiting the possibilities of analysing the interaction with the population. Nevertheless, their numerical conclusions provide an opportunity to confirm the importance of including boundary dynamics in such problems.

Pesticide pollution in rivers and their compartments has increased due to industrial discharge and excessive agricultural use. These residues contaminate water, sediments, and aquatic organisms, posing serious health risks to humans. Organochlorine pesticides such as DDT, HCH, endosulfan, etc., are the most commonly found. The study by A.K.Chopra *et al*. [18] outlined the classification and toxicity of pesticides, discussed alternative solutions, and emphasised raising public awareness about the issue. In their paper, J. Chung & O. Kwon [19] investigated two-species competition-diffusion systems with different intrinsic growth rates, carrying capacities, and dispersal strategies (random and Fokker-Planck diffusion). A general criterion for the global dynamics was established, along with conditions for coexistence and stability. The results highlighted the significant impact of heterogeneous competition strengths and dispersal behaviours on ecosystem stability.

W. Chen & Ya. Chen [20] considered the Lotka-Volterra competition model with cross-diffusion under homogeneous Dirichlet boundary conditions was considered, where cross-diffusion represents mutual avoidance between two species due to competition. Using the method of upper and lower solutions, sufficient conditions for the existence of positive solutions were established when the cross-diffusion coefficients are sufficiently small. Additionally, conditions for the nonexistence of positive solutions were also investigated.

Conclusions

This study presented a mathematical model for pollutant transport in a river environment, incorporating spatio-temporally variable diffusion and advection coefficients. The model also accounted for the ecological interaction between a biological population and toxicants, emphasising how population density influences, and is influenced by, the concentration of harmful substances. The graph depicting toxicant concentration $v(x, t)$ illustrated the temporal evolution of the contaminant distribution. Initially, at $t = 0$, the toxicant was sharply localised within a confined region. As time progresses to $t = 2.5$, the concentration profile became smoother and more diffuse, indicating the effect of dynamic diffusion and advection processes.

The presence of a sharp gradient at the initial time transitioning into a smooth concentration curve confirms the model's ability to simulate the natural spread of pollutants with a free boundary. This reflects the physical phenomenon whereby contaminants gradually invade previously unaffected regions due to variable transport properties. The results demonstrate that the use of space- and time-dependent coefficients leads to more realistic simulation of environmental processes, offering better insight into the spread behaviour of toxicants under fluctuating ecological conditions. Importantly, the simulations revealed that the contaminant does not necessarily spread throughout the entire river domain. This outcome supports the theoretical finding that population persistence is possible in uncontaminated zones, which is a significant ecological implication. Therefore, incorporating such models into environmental monitoring and pollution control strategies can improve predictions and guide more effective management decisions.

This study proposed a mathematical model to analyse the effect of environmental toxicants on population dynamics over time. The figure illustrates the temporal evolution of population density $u(x, t)$. Initially, at $t = 0$, the population density remained relatively low and declined further in regions where toxicant concentrations are higher. However, by $t = 2.5$, the population had significantly increased and reached a more stable distribution, suggesting recovery in areas where pollutant levels have decreased or become negligible.

The results confirm that the presence of toxicants directly affects the spatial distribution of the population. In particular, the observed population growth in regions where toxicant concentration diminished aligns with the model's coupled reaction-diffusion-advection mechanisms. This behaviour demonstrates the model's ability to accurately capture the complex ecological interactions between contamination and species dynamics. It also highlights the potential for population recovery when environmental conditions improve. Thus, the findings indicate that under favourable ecological conditions-such as pollutant reduction or optimised flow parameters-the population can recover and exhibit stable growth. This reinforces the practical relevance of the model for environmental monitoring and for designing effective pollution control and ecosystem restoration strategies.

Unlike static-coefficient models that oversimplify environmental dynamics, this approach reflects the realistic variability in river geometry, flow velocity, and ecological feedbacks. Furthermore, the integration of a free boundary to represent the moving front of the toxicant distinguishes this model from prior frameworks, allowing for a more accurate simulation of pollutant spread and retreat. While earlier works often focused on irreversible population decline under contamination, the current simulations demonstrate conditions under which population persistence and regrowth are possible once toxicant levels subside. Future research could extend this model by incorporating multi-species interactions to explore how different trophic levels respond to pollution gradients. Additionally, coupling the model with empirical field data would enable calibration and validation in real-world river systems, enhancing its predictive capabilities.

Acknowledgements

None.

Funding

None.

Conflict of Interest

None.

References

[1] Cosner Ch. Reaction-diffusion-advection models for the effects and evolution of dispersal. Discrete Contin Dyn Syst. 2014;34(5):1701–45. DOI: 10.3934/dcds.2014.34.1701
[2] Peng W, Salmaniw Yu, Wang X. Threshold dynamics of a reaction-diffusion-advection model with spatially varying parameters to analyze population persistence in river-like settings. J Math Biol. 2024;88:76. DOI: 10.1007/s00285-024-02097-6
[3] Lam K-Y, Lou Y. Introduction to reaction-diffusion equations: Theory and applications to spatial ecology and evolutionary biology. Heidelberg: Springer; 2022. 312 P. DOI: 10.1007/978-3-031-20422-7
[4] Fabre C, Sonke JE, Tananaev N, Teisserenc R. Organic carbon and mercury exports from pan-Arctic rivers in a thawing permafrost context – a review. Sci Total Environ. 2024;954:176713. DOI: 10.1016/j.scitotenv.2024.176713
[5] Tang D, Zhou P. On a Lotka-Volterra competition-diffusion-advection system: Homogeneity vs heterogeneity. J Differ Equations. 2020;268(4):1570–99. DOI: 10.1016/j.jde.2019.09.003
[6] Liu K, Tang D, Chen Sh. Evolution of dispersal in a stream with better resources at downstream locations. Stud Appl Math. 2025;154(2):e70017. DOI: 10.1111/sapm.70017
[7] Takhirov JO, Boborakhimova MI. On the mathematical model of the concentration of pollutants and their impact on the population of the river. Res Appl Math. 2024;21:100414. DOI: 10.1016/j.rinam.2023.100414
[8] Horstmann D, Winkler M. Boundedness vs. blow-up in a chemotaxis system. J Differ Equations. 2005;215(1):52–107. DOI: 10.1016/j.jde.2004.10.022
[9] Ladyzhenskaya OA, Solonnikov VA, Uraltseva NN. Linear and quasilinear equations of parabolic type. Vol. 23. Providence: American Mathematical Society; 1968. 648 P.
[10] Friedman A. Partial differential equations of parabolic type. Englewood Cliffs: Prentice Hall; 1964. 345 P.
[11] Naac VI, Naac IE. Mathematical models and numerical methods in problems of environmental monitoring of the atmosphere. Moscow: Fizmatlit; 2010. 328 P.
[12] Müller DF, Wibbing D, Herwig C, Kager J. Simultaneous real-time estimation of maximum substrate uptake capacity and yield coefficient in induced microbial cultures. Comput Chem Eng. 2023;173:108203. DOI: 10.1016/j.compchemeng.2023.108203
[13] Keller EF, Segel LA. Model for chemotaxis. J Theor Biol. 1971;30(2):225–34. DOI: 10.1016/0022-5193(71)90050-6
[14] Lin F, Ren H, Qin J, Wang M, Shi M, Li Y, et al. The influence of seasonal variability of flow parameters on pollutant transport in rivers with spatiotemporal diffusion coefficients. J Environ Manage. 2024;354:120314. DOI: 10.1016/j.jenvman.2024.120314
[15] Chen Q, Mei K, Dahlgren RA, Wang T, Gong J, Zhang M. Impacts of land use and population density on seasonal surface water quality using a modified geographically weighted regression. Sci Total Environ. 2016;572:450–66. DOI: 10.1016/j.scitotenv.2016.08.052

[16] Naizabayeva L, Sembina G, Suleimenova M, Manapova A. Modelling the distribution of atmospheric pollutants in the urban environment. Preprints. 2024. DOI: 10.20944/preprints202410.2551.v1
[17] Aksan EN, Karabenli H, Esen A. An application of finite element method for a moving boundary problem. Therm Sci. 2018;22(1):25–32. DOI:10.2298/TSCI170613268A
[18] Chopra AK, Sharma MK, Chamoli S. Bioaccumulation of organochlorine pesticides in aquatic system – an overview. Environ Monit Assess. 2011;173:905–16. DOI: 10.1007/s10661-010-1433-4
[19] Chung J, Kwon O. Dynamics of Lotka-Volterra competition systems with Fokker-Planck diffusion. J Funct Spaces. 2018. DOI: 10.1155/2018/7879598
[20] Chen W, Chen Ya. A Lotka-Volterra competition model with cross-diffusion. Abstr Appl Anal. 2013. DOI: 10.1155/2013/624352

Дарыялардагы популяция жана булгоочу заттардын динамикасы үчүн өзгөрүлмө диффузия жана адвекция коэффициенттери менен сызыктуу эмес эркин чек ара модели

Махбуба Боборахимова
Физика-математика илимдеринин кандидаты, илимий кызматкер
В.И. Романовский атындагы Математика институту,
Өзбекстан Республикасынын Илимдер Академиясы
100174, Университет көч., 9-үй, Ташкент ш., Өзбекстан Республикасы
https://orcid.org/0009-0001-3446-7624

Орзигул Пардаева
Илимий кызматкер
В.И. Романовский атындагы Математика институту,
Өзбекстан Республикасынын Илимдер Академиясы
100174, Университет көч., 9-үй, Ташкент ш., Өзбекстан Республикасы
https://orcid.org/0009-0008-7317-1229

Аннотация. Табигый суу чөйрөлөрүндө булгоочу заттардын таралуу ылдамдыгын мүнөздөгөн диффузия коэффициенти менен суунун агымы аркылуу ташылышын чагылдырган адвекция коэффициенти чоң мейкиндиктик жана убакыттык өзгөрмөлүүлүккө ээ. Бул өзгөрүүлөр дарыянын геометриясынын, агым ылдамдыгынын, температуранын жана сезондук динамиканын өзгөрүшү менен шартталган. Бул татаалдыктарды толук эске алуу үчүн, бул изилдөөдө диффузия жана адвекция коэффициенттеринин мейкиндиктик-убакыттык өзгөрмөлөрүн камтыган өркүндөтүлгөн моделдик түзүм сунушталган. Бул коэффициенттер популяциянын жыштыгына жана чөйрөдөгү уулуу заттардын концентрациясына жараша өзгөрөт деп болжолдонгон, бул булгоочу заттардын ташылыш процесстерин реалдуураак чагылдырууга мүмкүндүк берет. Бул макалада уулуу заттардын таралышын чагылдырган эркин чек аралуу туунду дифференциалдык теңдемелердин (PDE) татаал системасы иштелип чыккан. Модель дарыядагы биологиялык популяция менен токсиканттын өз ара аракетин экологиялык жана гидродинамикалык факторлорду эске алуу менен сүрөттөйт. Чечимдин регулярдуулугун камсыз кылуу үчүн популяциянын жыштыгын $u(x,t)$, токсиканттын концентрациясын $v(x,t)$ жана эркин чек аранын абалын $s(t)$ камтыган априордук эсептөөлөр киргизилет, ошондой эле Гёльдер туташтыгы боюнча баалоолор жүргүзүлөт. Классикалык чечимдердин глобалдуу бар экендиги жана жалгыздыгы Лере-Шаудердин кыймылсыз чекит жөнүндөгү теоремасы жана энергияга негизделген ыкмалар аркылуу катуу далилденет. Айрым параметрдик режимдерде токсикант дарыянын бардык аянтына тарай албастыгы аныкталды, бул өз кезегинде популяцияга жабыркабаган аймактарда жашап калууга мүмкүндүк берди. Эркин чек аралуу татаал туюнтулган туура эмес маселени аналитикалык жактан изилдөө кыйындыгына байланыштуу аныкталбаган (неявный) сандык схемалар колдонулду. Python тилинде ишке ашырылган сандык эксперименттер жана графикалык визуализациялар теориялык жыйынтыктарды ырастап, экологиялык параметрлер менен булгоочу заттардын динамикасынын өз ара аракетин чагылдырды. Алынган натыйжалар ар түрдүү чөйрөлүк шарттардын биологиялык популяциялардын туруктуулугуна жана уулуу заттардын мейкиндик-убакыттык эволюциясына кандайча таасир этерин көрсөтөт

Негизги сөздөр: татаал динамика; булгоочу заттардын таралышы; эркин чек ара маселеси; сандык моделдөө; диффузия коэффициенти

Нелинейная модель со свободной границей с переменными коэффициентами диффузии и адвекции для динамики популяции и загрязнителей в реках

Махбуба Бобораохимова
Кандидат физико-математических наук, научный сотрудник
Институт математики им. В.И. Романовского Академии наук Республики Узбекистан
100174, ул. Университетская, 9, г. Ташкент, Республика Узбекистан
https://orcid.org/0009-0001-3446-7624

Орзигул Пардаева
Научный сотрудник
Институт математики им. В.И. Романовского Академии наук Республики Узбекистан
100174, ул. Университетская, 9, г. Ташкент, Республика Узбекистан
https://orcid.org/0009-0008-7317-1229

Аннотация. В естественных водных средах как коэффициент диффузии, характеризующий скорость рассеивания загрязняющих веществ, так и коэффициент адвекции, описывающий перенос за счет потока воды, демонстрируют значительную пространственно-временную изменчивость. Эти изменения обусловлены изменениями в геометрии реки, скорости течения, температуре и сезонной динамике. Чтобы лучше охватить эти сложности, в этом исследовании представлена усовершенствованная модельная структура, которая включает пространственно-временные переменные коэффициенты диффузии и адвекции. Предполагалось, что эти коэффициенты зависят как от плотности популяции, так и от концентрации токсикантов окружающей среды, что позволяет более реалистично представить процессы переноса загрязняющих веществ. В этой статье разрабатывалась нелинейная система уравнений в частных производных (PDE) со свободной границей для представления динамического аспекта рассеивания токсичных веществ. Модель характеризует взаимодействие между речной биологической популяцией и токсикантом, учитывая экологические и гидродинамические влияния. Для обеспечения регулярности решения устанавливаются априорные вычисления, включая плотность популяции $u(x, t)$, концентрацию токсиканта $v(x, t)$ и положение свободной границы $s(t)$, а также оценки непрерывности Гельдера. Глобальное существование и единственность классических решений строго доказаны с помощью теоремы Лере-Шаудера о неподвижной точке и методов, основанных на энергии. Были выявлены режимы параметров, при которых токсикант не мог распространиться по всей площади реки, тем самым позволяя популяции выживать в незатронутых областях. Из-за аналитической сложности нелинейной задачи свободной границы для моделирования использовались неявные численные схемы. Численные эксперименты, реализованные на Python с графическими визуализациями, подтверждают теоретические результаты и иллюстрируют взаимодействие между экологическими параметрами и динамикой загрязняющих веществ. Полученные результаты показывают, как различные условия окружающей среды влияют на устойчивость биологических популяций и пространственно-временную эволюцию концентраций токсичных веществ

Ключевые слова: нелинейная динамика; распространение загрязняющих веществ; задача со свободной границей; численное моделирование; коэффициент диффузии

VOL. 4, No. 1 Journal of Osh State University. Mathematics. Physics. Technical Sciences 2025

Journal of Osh State University Journal homepage: https://mpht-oshsu.com/en

UDC 662.997-534-4
DOI: 10.52754/16948645_2025_4(1)_61

Article's History:
Received 11.02.2025 Revised 14.05.2025 Accepted 19.06.2025

Analysis and research of solar heating in the design of residential buildings

Tugolbay Matisakov*
PhD in Technical Sciences, Associate Professor
Osh State University
715000, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0000-0001-8644-059X

Zhumakadyr Matisakov
PhD in Technical Sciences, Associate Professor
Osh Technological University
723503, 81 N. Isanov Str., Osh, Kyrgyz Republic
https://orcid.org/0009-0000-7643-5324

Abstract. The aim of the study was to develop a mathematical model for improving the energy efficiency of residential buildings through the seasonal use of solar heating. A method for the theoretical calculation of energy-efficient houses was proposed, taking into account the geometric parameters of windows and the design features of the window roof. The conditions for the optimal placement of windows for effective capture of solar radiation during the heating season were identified. The study paid particular attention to the climatic characteristics of Kyrgyzstan, such as the duration of the heating season, the level of solar radiation and the potential for reducing the consumption of traditional energy sources. A climate analysis of the cities of Osh and Bishkek showed that even on the shortest winter days, it is possible to obtain a significant amount of solar energy, sufficient for partial or complete heating of premises. A mathematical model of heat loss has been developed, taking into account the temperature difference between the indoor and outdoor environments, as well as the heat transfer coefficient of the building envelope. This allows for an assessment of the duration of effective use of solar energy for heating. Key design parameters have been formalised, including the angle of incidence of sunlight, the length of the roof overhang, the height of the window and the geometry of the facade. Formulas for calculating the length and height of the canopy, taking into account seasonal changes in the position of the sun, have also been proposed. The article presents a roof and window layout that provides protection from overheating in summer and maximum solar energy inflow in winter. The study confirms that well-designed solar heating systems can significantly reduce the load on central heating and increase the efficiency of renewable energy use. The presented methods are applicable in the design of modern energy-efficient buildings, especially in regions with mountainous terrain and a long heating season. Thus, this study is of great importance for the practical implementation of solar heating systems capable of ensuring the sustainable and efficient use of solar energy, taking into account local climatic characteristics

Keywords: renewable energy sources; thermal efficiency; architectural solutions; thermal insulation of buildings; climatic characteristics; optimisation of building elements; seasonal adaptation of structures

Introduction

Heating residential buildings is a pressing issue, as the constant rise in the cost of coal and other traditional energy sources, as well as the environmental problems associated with their use, require ways to reduce consumption. One way to solve this problem is to expand the use of solar energy, including its application in heating systems. Solar energy is a renewable resource that can significantly reduce dependence on

Suggested Citation:
Matisakov T, Zhumakadyr M. Analysis and research of solar heating in the design of residential buildings. J Osh State Univ Math Phys Tech Sci. 2025;4(1):61–9. DOI: 10.52754/16948645_2025_4(1)_61

*Corresponding author

centralised energy sources and reduce the load on heating infrastructure. Given the climatic characteristics of Kyrgyzstan and the potential for using solar heating in mountainous areas, research in this field is important for improving the energy efficiency of residential buildings.

When using solar energy in passive heating systems, it is necessary to study the location of windows, which is an important element of a residential building. Analysing the impact of the geometric parameters of windows and roofs on solar energy capture, as well as developing mathematical models to optimise these parameters, will help create effective solutions that can be applied in real-world conditions, especially in regions such as Osh and Bishkek.

In their study, scientists R.A. Akparaliev *et al*. [1] analysed the climatic conditions of Kyrgyzstan in detail, creating a resource map that includes geographical coordinates, administrative divisions, as well as solar radiation parameters and surface inclination angles. Their work made it possible to take into account the characteristics of the local climate when designing solar heating systems, but issues of seasonal adaptation of structures and integration of geometric characteristics of buildings remained insufficiently addressed. According to reports from international energy organisations, heating buildings accounts for 20% to 30% of total final energy consumption, and in countries with cold climates, this figure can reach 40% or more [2]. The book by N.R. Avezova *et al.* [3] examined the energy efficiency of residential buildings with an emphasis on the use of the Trombe wall passive solar system. However, the authors do not conduct a detailed analysis of architectural solutions in a broader context, such as building shape, orientation, planning features, and other parameters that affect energy consumption reduction.

The study by S.I. Khamraev [4] noted the importance of improving the energy efficiency of residential heating systems through the use of renewable energy sources, in particular solar energy. The author proposed a combined heating scheme that allows up to 70% of the heat load to be covered during the winter period through the effective use of solar collectors. The work was based on experimental research conducted in real conditions in the south of Uzbekistan, in the city of Karshi, from November 2020 to February 2021, which gives the results practical reliability and relevance. The proposed system takes into account the climatic characteristics of the region, which is characterised by high levels of solar radiation and a continental climate, and demonstrates the technical feasibility of solar heating in domestic conditions.

However, despite the high practical value of the study, the work lacks an in-depth analysis of architectural solutions for buildings, such as orientation, thermal insulation, insolation, building envelope materials, etc., which have a significant impact on reducing the overall heat load. Failure to take these factors into account narrows the overall systemic assessment of energy efficiency and limits the adaptability of the proposed model to a wider range of architectural contexts.

Sh. Wang *et al*. [5] noted that energy-active houses provide significant savings in energy costs by using solar energy for heating and lighting. The authors emphasised that such houses contribute to reducing the carbon footprint, as they do not depend on fossil fuels, and reduce dependence on external electricity supplies, providing greater autonomy. The Sunny Inside project, presented at the Solar Decathlon China 2013 competition [5], developed and analysed key design elements such as an eco-friendly atrium, shading systems, natural ventilation, a heat storage system and thermal insulation. However, the project does not take into account the specific climatic conditions of Kyrgyzstan, which limits its applicability in this region. This article proposes an adaptation of the Sunny Inside design elements to take into account the climatic characteristics of Kyrgyzstan, which is an important step towards improving the energy efficiency of residential buildings in the country.

Given that previous studies and implemented projects in the field of solar heating did not fully take into account the specific features of the region and the architectural parameters of buildings, this study set out to develop a mathematical model that would improve the energy efficiency of residential buildings through the rational seasonal use of solar energy. The proposed theoretical approach is based on adapting existing technologies to the specific conditions of Kyrgyzstan and includes taking into account the geometry of the glazing, the design features of the roof and other factors affecting heat loss.

Materials and Methods

This study examined the optimisation of structural elements of glazing and roofing for more efficient use of solar energy throughout the year. A mathematical approach based on geometric and climatic parameters was used as a basis, allowing for a quantitative assessment of insolation through window openings. To refine the insolation parameters under clear sky conditions, this study used a simplified model of direct and diffuse solar radiation proposed by R.E. Bird & R.L. Hulstrom [6]. This model made it possible to quantitatively assess the amount of solar energy reaching horizontal and vertical surfaces at different times of the year, taking into account the sun's altitude, atmospheric transparency, and climatic conditions in the region. This approach made it possible to analyse light transmission and the formation of conditions conducive to improving the thermal efficiency of building envelopes in different seasons. Key factors were taken into account: the angle of incidence of sunlight, the geometry of the building, the size of the windows, the angle of the canopy and the duration of solar radiation [7]. One of the most important parameters of solar radiation is the angle of incidence of sunlight on the surface, which determines the proportion of direct solar radiation passing through the windows of an energy-efficient house [8]. During the heating season, sunlight may not fall directly into the window, encountering obstacles such as trees, rocks or hills that temporarily block direct solar radiation.

Determining the optimal angle of incidence of sunlight is the initial stage of design. This angle changes throughout the year and is determined by the ratio of the sun's height above the horizon to the distance from the Earth to the Sun. Formally, the angle of incidence θ_s is expressed as [9]:

$$\theta_s = arcsin\left(\frac{h_s}{l_r}\right), \tag{1}$$

where h_s – height the sun above the horizon, l_r – the distance from the Sun to the Earth. Let the height of the window be h. In order for the sun's rays to hit the window in winter, the horizontal projection of the rays must cross the upper edge of the window. In summer, the roof must be long enough to cover the upper part of the window and prevent overheating. The length of the roof L is calculated using the formula:

$$L = h \cdot tg(\theta_s), \tag{2}$$

where L is the length of the visor blocking sunlight. The distance from the upper edge of the window to the lower edge of the roof (h_k) depends on the length of the visor and the angle of incidence of the rays:

$$h_k = L \cdot sin(\theta_s). \tag{3}$$

These formulas made it possible to calculate the optimal sizes and angles for windows and roofs, ensuring efficient use of solar energy in different seasons. For climate analysis, data on the duration of the heating season and total solar radiation were used, obtained using specialised Delphi 7.0 software. The Python programming language was used to implement the mathematical model and perform numerical calculations, including modelling indoor temperature dynamics and constructing graphs, which ensured flexibility and accuracy of calculations and allowed for the automation of the results visualisation process. The indoor heat exchange model was described by a differential equation:

$$\frac{d_t}{d_\tau} = -k(t - t_c), \tag{4}$$

where t is the indoor temperature at time τ, t_c is the outdoor air temperature, and k is the heat transfer coefficient of the walls. The solution to the equation with initial conditions $\tau = 0$, $t = t_0$ is:

$$t(\tau) = t_c + (t_o - t_c) \cdot e^{-k\tau}. \tag{5}$$

This made it possible to model the temperature dynamics inside the room in the absence of additional heating, taking into account heat loss through the walls.

Results and Discussion

Ordinary window glass transmits about 3-4% of ultra-shortwave radiation (280-315 nm) and almost completely blocks harmful ultraviolet radiation in the range of 100-280 nm. At the same time, it transmits up to 75% of less dangerous ultraviolet radiation (315-400 nm), which not only contributes to the accumulation of heat in the room, but also has an antiseptic effect, destroying harmful microbes [10].

Since windows are key elements through which solar heat enters energy-efficient buildings, an important stage in the design process is determining their optimal location [11]. The main principle here is the rational use of sunlight, taking into account seasonal changes in the position of the sun on the horizon [12]. Accordingly, the height of window openings and the structural placement of the roof must be adapted to maximise the intake of solar energy during the heating season – autumn, winter and spring [13].

For the effective use of solar radiation, it is necessary to take into account the dynamics of the angle of incidence of sunlight in different seasons. At the beginning of the heating season, the angle of incidence is between 27° and 40°, and in the winter months, the sun's rays are almost parallel to the horizon, which makes south-facing windows particularly effective for natural heating of rooms. However, during the transition periods before the start of the heating season, there may be an excessive amount of solar radiation, which can lead to overheating of interior spaces. This requires careful design of architectural elements, in particular the placement of windows, overhangs and canopies, as described by T. Muneer *et al*. [14].

Calculations obtained using a simplified model of direct and diffuse solar radiation were used to determine the rational geometry of windows and canopies, aimed at limiting solar overheating in summer and maximising heat gain in winter. For example, if the width of the window is 1.5 m and the height is 1.7 m, the optimal length of the canopy should be about 1.4 m, and the distance from the upper edge of the window to the lower plane of the canopy should be about 1.06 m. With these parameters, summer sunlight will be completely blocked, preventing overheating, while in winter, solar radiation will freely enter the room, providing natural heating. To visualise the geometric layout of the windows and roof, the authors developed the diagram shown in Figure 1.

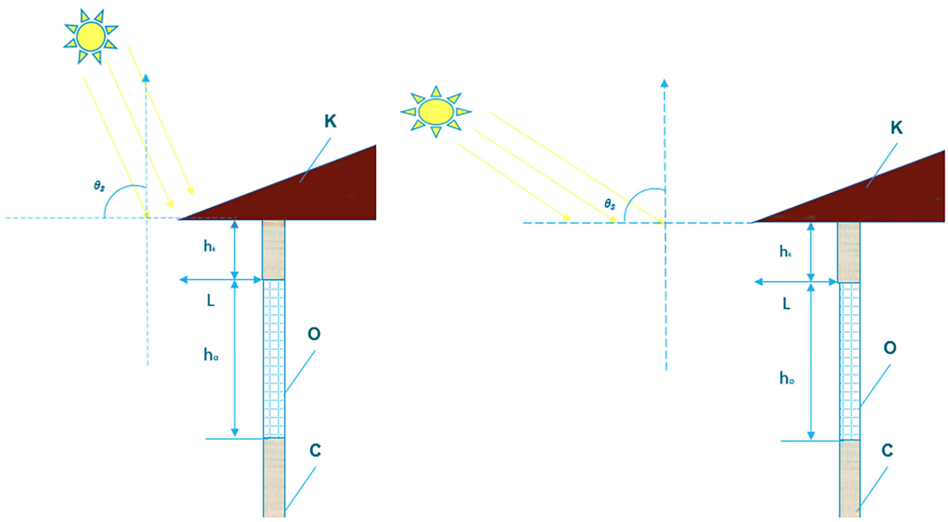

Figure 1. Scheme of the optimal arrangement of the roof and windows of the building
Notes: O – windows oriented towards the sun; K – roof optimised for the heating season; C – building walls
Source: developed by the authors

As a result of the modelling, a geometric diagram of the optimal location of windows and the roof of the building was developed (Fig. 1), taking into account the seasonal characteristics of solar radiation. Based on this diagram, an analysis of insolation for the summer and winter periods was carried out. In summer (Fig. 1a), the roof is designed to shade the windows and prevent overheating of the rooms, while in winter (Fig. 1b), the design allows maximum use of solar heat for natural heating of the interior spaces. This approach ensures effective management of solar energy throughout the year and helps to reduce energy costs for air conditioning and heating.

This method is particularly relevant for mountainous regions such as Kyrgyzstan, where the long heating season and high solar energy potential create favourable conditions for improving the energy efficiency of residential buildings and reducing the consumption of traditional energy sources. In such conditions, it is particularly important to accurately assess the duration of the heating period and the characteristics of solar radiation, which allows for the optimisation of heating system design using passive and active solar technologies. Figure 2 shows the dynamics of the duration of the heating season (in days) in the city of Bishkek over the last 15 years.

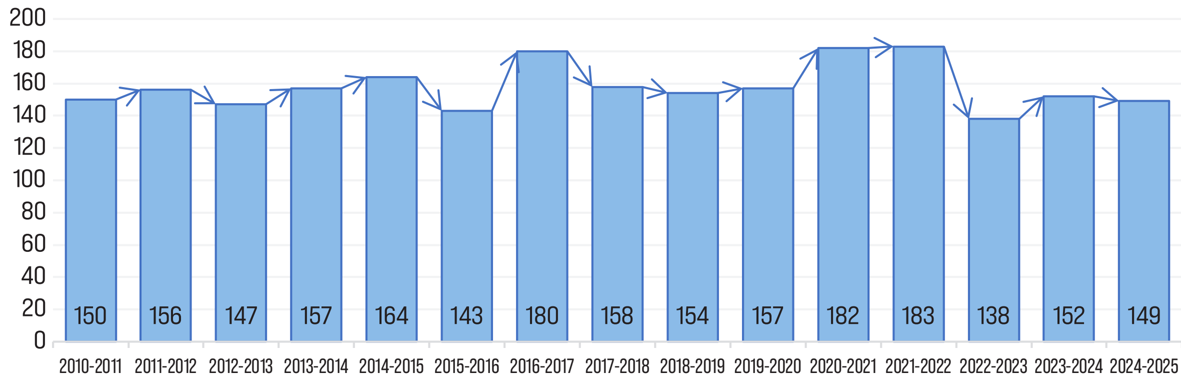

Figure 2. Duration of the heating season (days) in Bishkek
Source: compiled by the authors based on [15]

The analysis shows that the duration of the heating season in Bishkek ranges from 138 to 183 days, with an average of about 158 days. Such significant seasonal fluctuations directly affect the choice and configuration of heating systems, as well as the calculation of thermal insulation and solar collector parameters. The duration of the heating season in other large cities of Kyrgyzstan, such as Osh, is close to that of Bishkek, as confirmed by a comparative analysis of data for several years [15]. This indicates the need to develop universal solutions that are adaptable to local conditions and take into account regional climate and terrain specifics.

To gain a more complete understanding of the region's solar potential, an assessment of daylight hours and total solar radiation was carried out using specialised software. For example, the shortest day of the heating season – 22 December – in the city of Osh lasts only 6 hours and 10 minutes, with total solar radiation of about 388.73 W·h/m^2 (Fig. 3). These data highlight the need to integrate seasonal changes in sunlight into energy consumption models and the design of energy-efficient heating systems.

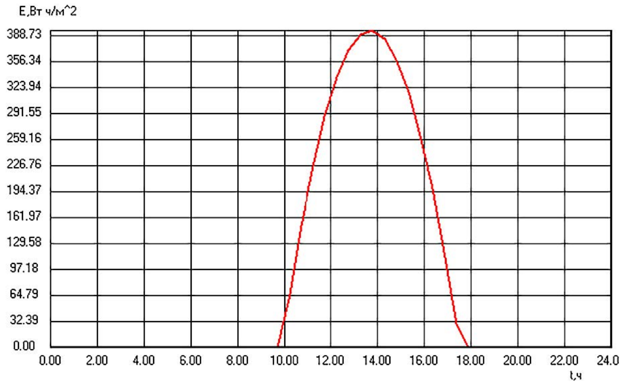

Figure 3. Forecast of total solar radiation for 22 December – the shortest day of the heating season (Osh)
Source: developed by the authors using Delphi 7.0

According to calculations, even on the shortest winter day, a significant amount of solar energy enters through a 4 m^2 window – approximately 9,573.24 W·h. This highlights the potential of solar energy as an important source of heat for energy-efficient buildings in cold climates. Using a mathematical model that takes into account an outside temperature of -5 °C and no additional heating, the dynamics of indoor temperature changes were simulated (Fig. 4).

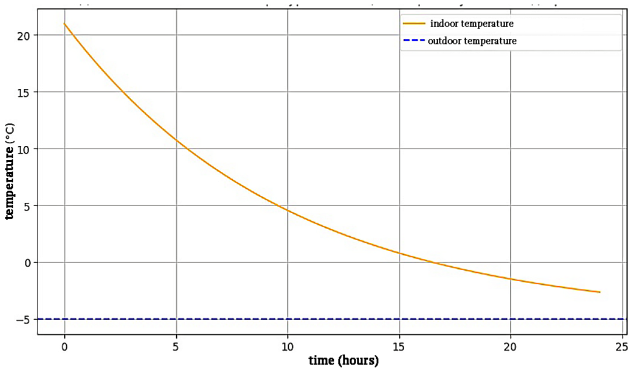

Figure 4. Temperature dynamics in a room without heating
Source: the authors' own calculations using Python

As can be seen from the results, a comfortable indoor temperature can be maintained for approximately 5 hours thanks to accumulated solar heat. After that, the temperature begins to gradually decrease, indicating the need to connect additional heating systems to maintain comfortable conditions. At the same time, the indoor temperature does not drop to the outdoor level (below 0 °C) for 15 hours, which significantly extends the time period during which energy costs for heating are reduced. Considering that the average time until sunrise in winter is about 16 hours, the use of solar energy in combination with properly designed architectural elements (windows, canopies, etc.) can reduce the consumption of traditional energy sources for heating by approximately 40%.

These results are consistent with preliminary calculations of insolation, modelling of the angles of incidence of sunlight and analysis of the duration of the heating season, which confirms the complexity and integrity of the proposed approach. Taken together, this demonstrates the high effectiveness of the proposed model for improving the energy efficiency of buildings in mountainous regions with long heating seasons, such as Kyrgyzstan. It should be noted that the proposed approach is consistent in a number of cases with the research of other scientists in the field of energy-efficient building design. For example, in an article on solar architecture, authors L. Zhong *et al.* [16] emphasised the importance of correct window orientation and the use of solar energy for heating in winter.

Nevertheless, there are a number of differences between the proposed model and other approaches. For example, the article considers a specific geographical area – Kyrgyzstan – and takes into account the natural features of this region, such as the duration of the heating season and the level of solar radiation. The research by K. Mehta *et al.* [17] specifically takes into account the climatic characteristics and conditions of different regions, which is particularly important as climate significantly affects the efficiency of solar heating systems. At the same time, many studies conducted in other countries focus on their own climatic conditions, which can lead to differences in the results of calculations. For example, studies conducted in regions with higher levels of solar radiation often use increased values of solar radiation and climatic parameters in their calculations, which leads to differences in recommendations for the design of solar heating systems. As noted by W. Mo *et al.* [18], climatic conditions, especially the level of solar radiation, significantly affect the design and efficiency of passive solar systems, requiring the adaptation of technical solutions to a specific region.

However, despite these differences, the conclusions of this article are conceptually very close to the results of other studies proposing the use of mathematical models to optimise solar architectures. In particular, the International Energy Agency (IEA) report [2] considered the use of energy for heating in conditions of short daylight hours, including the shortest days of the year. The document emphasises that the amount of solar energy available in the winter months depends significantly on geographical location and local climatic conditions. This is confirmed by the calculations made in the article, where on 22 December, the shortest day of the heating season, 388.73 W·h/m^2 of solar energy is available. Other studies, for example, those conducted in the United Kingdom and Scandinavian countries, note that the results may vary significantly due to lower levels of solar radiation associated with longer winters and a low angle of incidence of sunlight. As pointed out by R. Renaldi & D. Friedrich [19] and M. Herrando & C.N. Markides [20], climatic conditions significantly affect the efficiency of solar heating systems and require design solutions to be adapted to low levels of insolation.

A. Olgyay & V. Olgyay [21] used a method for calculating solar insolation, which facilitated their adaptation to different types of buildings and climatic conditions. This is particularly relevant for architects and designers who need to consider not only the geographical orientation of the building but also seasonal changes in the angle of incidence of sunlight. In particular, the equations proposed by the authors allow determining the optimal parameters of architectural elements: the length of the canopy blocking direct solar radiation in summer and the height of the glazing ensuring the penetration of sunlight in winter. Such calculations serve as a practical guide for the design of energy-efficient buildings.

At the same time, there are several aspects that could be further explored. For example, the influence of different types of window glass on the efficiency of solar heating could be considered. The article mentions that glass transmits up to 75% of ultraviolet radiation in the 315-400 nm range, but does not consider other materials that could increase the efficiency of solar radiation. Thus, in a classic work by B.Y.H. Liu & R.C. Jordan [22], empirical relationships were proposed for calculating scattered solar radiation on a horizontal surface based on total radiation data. These relationships allow determining both instantaneous and average daily values of scattered radiation for various weather conditions, including clear and cloudy days, and are widely used in modelling solar heat exchange in buildings.

The proposed model is useful for practical application, especially in countries with variable climates, such as Kyrgyzstan. It provides clear recommendations for the design of roofs and windows for optimal use of solar energy, which can significantly improve the energy efficiency of buildings and reduce the need for artificial heating. The work confirms the importance of competent solar architecture design, which is supported by other studies, and opens up prospects for the application of such methods in real conditions, including their adaptation to specific climatic conditions.

Conclusions

Even on the shortest winter days, the amount of solar radiation can provide a significant amount of solar energy, sufficient to heat rooms through windows. This allows for a significant reduction in dependence on traditional energy sources such as coal and gas. To assess the efficiency of solar energy use in heating, it is important to consider heat transfer through the walls of the building and the temperature difference between the interior and exterior environments. Mathematical modelling of these processes allows for accurate calculation of the time required for solar energy to provide the necessary heating for a room.

The results confirmed that even on the shortest winter days, a significant amount of solar energy can be obtained through correctly oriented and well-designed windows. The heat loss model made it possible to estimate the duration of effective solar energy use and its impact on the internal temperature of rooms. The study showed that competent design of solar heating systems (taking into account the angle of the roof, the orientation of the windows and the length of the eaves) can significantly reduce heat loss and reduce the load on traditional energy sources. This is particularly relevant for regions with long heating seasons and high solar insolation, such as Kyrgyzstan.

Thus, the results of the study confirm the importance of using solar heating to improve the energy efficiency of residential buildings in Kyrgyzstan. The theoretical and computational approaches developed provide a solid foundation for the further development and implementation of solar technologies in architectural design in the region. The use of solar energy in heating systems is a promising and effective way to solve energy shortages and environmental problems, especially in Kyrgyzstan with its natural characteristics and long heating seasons.

Acknowledgements

None.

Funding

None.

Conflict of Interest

None.

References

[1] Akparaliev RA, Mederov TT, Obozov ADzh, Ashimbekova B. Analysis of solar radiation data to create a resource map. Sci Educ Eng. 2022;2(74):29–35. DOI: 10.54834/16945220_2021_2_29

[2] International Energy Agency. Space heating [Internet]. [cited 2023 September]. Available from: https://www.iea.org/reports/space-heating

[3] Avezova NR, Avezov RR, Samiev KA, Kakharov SK. Comparative heating performance and engineering economic indicators of the "trombe wall" system in different climate zones of Uzbekistan. Appl Sol Energy. 2021;57(2):128–34. DOI: 10.3103/S0003701X21020031

[4] Khamraev SI. Study of the combined solar heating system of residential houses. In: Proceedings of the II international conference on current issues of breeding, technology and processing of agricultural crops, and environment (CIBTA-II-2023). Vol. 71. Les Ulis: EDP Sciences; 2023. P. 1–8. DOI: 10.1051/bioconf/20237102017
[5] Wang Sh, Shi F, Zhang B, Zheng J. The passive design strategies and energy performance of a zero-energy solar house: Sunny inside in solar decathlon China 2013. J Asian Archit Build Eng. 2018;15(3);543–8. DOI: 10.3130/jaabe.15.543
[6] Bird RE, Hulstrom RL. Simplified clear sky model for direct and diffuse insolation on horizontal surfaces. Golden (CO): Solar Energy Research Institute; 1981. 57 P.
[7] Bastien D, Athienitis AK. Methodology for selecting fenestration systems in heating dominated climates. Appl Energy. 2015;154:1004–19. DOI: 10.1016/j.apenergy.2015.05.083
[8] Faye I, Ndiaye A, Mamadou E. Influence of the incidence angle modifier and radiation on the performance of textured and nontextured monocrystalline silicon solar cells. In: Solar cells: Theory, materials and recent advances. London: IntechOpen; 2021. Chapter 8. DOI: 10.5772/intechopen.96160
[9] Duffy J, Beckman WA. Solar engineering of thermal processes. 2ed ed. Hoboken: Wiley-Interscience; 1991. 926 P.
[10] Obozov ADzh, Botpaev RM. Renewable energy sources: Textbook for universities. Bishkek: Kyrgyz State Technical University; 2010. 224 P.
[11] Hudson Architects. Free energy: optimising orientation and glazing for solar gains [Internet]. [cited 2025 Jul 7]. Available from: https://hudsonarchitects.co.uk/journal/architectural-insights/free-energy-optimising-orientation-and-glazing-for-solar-gains/
[12] Iqbal M, Liu BYH. A model for estimating solar radiation on tilted surfaces using hourly data. Sol Energy. 1980;24(1):37–46.
[13] Gueymard CA. The sun's total and spectral irradiance for solar energy applications and solar radiation models. Sol Energy. 2004;76(4):423–53. DOI: 10.1016/j.solener.2003.08.039
[14] Muneer T, Gueymard C, Kambezidis H. Solar radiation and daylight models. 2nd ed. London: Routledge; 2004. 392 P. DOI: 10.4324/9780080474410
[15] When heating was turned on in Bishkek over the last 18 years – schedule [Internet]. [cited 2022 October 25]. Available from: https://ru.sputnik.kg/20221025/bishkek-otoplenie-sezon-nachalo-grafik-1069326818.html
[16] Zhong L, Wu D, Zhang B, Zhang Y, Liang X. Study on the impact of design parameters of photovoltaic combined vacuum glazing (PVCVG) on the energy consumption of buildings in Lhasa. Build. 2025;15(4):649. DOI: 10.3390/buildings15040649
[17] Mehta K, Ehrenwirth M, Trinkl C, Zörner W, Greenough R. A parametric study on the feasibility of solar-thermal space heating and hot water preparation under cold climates in Central Asian rural areas. In: Proceedings of the 13th international conference on solar energy for buildings and industry (EuroSun 2020). Brussels: International Solar Energy Society (ISES); 2020. DOI: 10.18086/eurosun.2020.04.03
[18] Mo W, Zhang G, Yao X, Li Q, DeBacker BJ. Assessment of passive solar heating systems' energysaving potential across varied climatic conditions: The development of the passive solar heating indicator (PSHI). Build. 2024;14(5):1364. DOI: 10.3390/buildings14051364
[19] Renaldi R, Friedrich D. Technoeconomic analysis of a solar district heating system with seasonal thermal storage in the UK. Appl Energy. 2019;236:388–400. DOI: 10.1016/j.apenergy.2018.11.030
[20] Herrando M, Markides CN. Hybrid PVT systems for domestic heat and power in the UK: Technoeconomic analysis. Appl Energy. 2016;161:512–32. DOI: 10.1016/j.apenergy.2015.09.025
[21] Olgyay A, Olgyay V. Solar control and shading devices. Princeton: Princeton University Press; 1957. 201 P. DOI: 10.1002/qj.49708436029
[22] Liu BYH, Jordan RC. The interrelationship and characteristic distribution of direct, diffuse and total solar radiation. Sol Energy. 1960;4(3):1–19. DOI: 10.1016/0038-092X(60)90062-1

Турак жай имараттарын долбоорлоодо күн жылытуусун анализдөө жана изилдөө

Түгөлбай Матисаков
Техника илимдеринин кандидаты, доцент
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0000-0001-8644-059X

Матисаков Жумакадыр
Техника илимдеринин кандидаты, доцент
Ош технологиялык университети
723503, Н. Исанов көч., 81, Ош ш., Кыргыз Республикасы
https://orcid.org/0009-0000-7643-5324

Аннотация. Изилдөөнүн максаты күн энергиясын сезондук колдонуу аркылуу турак жай имараттарынын энергия эффективдүүлүгүн жогорулатуунун математикалык моделин иштеп чыгуу болгон. Терезелердин геометриялык параметрлерин жана терезе чатырынын конструкциялык өзгөчөлүктөрүн эске алуу менен энергия-активдүү үйлөрдү долбоорлоо үчүн теориялык эсептөө методу сунушталат. Жылытуу мезгилинде күн радиациясын эффективдүү кармоо үчүн терезелерди оптималдуу жайгаштыруу шарттары аныкталган. Изилдөөдө Кыргызстандын климаттык өзгөчөлүктөрүнө өзгөчө көңүл бурулган, мисалы, жылытуу мезгилинин узактыгы, күн радиациясынын деңгээли жана салттуу энергия булактарын керектөөнү кыскартуу потенциалы. Ош жана Бишкек шаарларынын климаттык анализи кыштын эң кыска күндөрүндө да жайларды жарым-жартылай же толук жылытуу үчүн жетиштүү күн энергиясын алууга болоорун көрсөттү. Ички жана тышкы чөйрөнүн ортосундагы температуранын айырмасын, ошондой эле курчап турган конструкциялардын жылуулук өткөрүмдүүлүк коэффициентин эске алган жылуулук жоготуусунун математикалык модели иштелип чыккан. Бул жылытуу үчүн күн энергиясын натыйжалуу пайдалануу узактыгын баалоого мүмкүндүк берет. Дизайндын негизги параметрлери, анын ичинде күндүн түшүү бурчу, чатырдын ашкан узундугу, терезенин бийиктиги жана фасаддын геометриясы расмий түрдө бекитилген. Күндүн абалынын сезондук өзгөрүшүн эске алуу менен чатырдын узундугун жана бийиктигин эсептөө үчүн формулалар да сунушталган. Макалада жайкысын ысып кетүүдөн коргоону жана кышында күн энергиясын максималдуу тийүүсүн камсыз кылган чатырларды жана терезелерди жайгаштыруунун схемасы келтирилген. Изилдөө жакшы долбоорлонгон күн жылытуу системалары борборлоштурулган жылытууга болгон жүктү олуттуу түрдө азайтып, энергиянын кайра жаралуучу булактарынын натыйжалуулугун жогорулата аларын тастыктайт. Сунушталган ыкмалар заманбап энергияны үнөмдөөчү имараттарды долбоорлоодо, өзгөчө тоолуу аймактарда жана жылытуу мезгили узак болгон аймактарда колдонулат. Ошентип, бул изилдөө жергиликтүү климаттык өзгөчөлүктөрдү эске алуу менен күн энергиясын туруктуу жана натыйжалуу пайдаланууну камсыз кыла ала турган күн жылытуу системаларын практикалык ишке ашыруу үчүн чоң мааниге ээ

Негизги сөздөр: энергиянын кайра жаралуучу булактары; жылуулук эффективдүүлүгү; архитектуралык чечимдер; имараттарды жылуулуктан коргоо; климаттык өзгөчөлүктөр; курулуш элементтерин оптималдаштыруу; структуранын сезондук ыңгайлашуусу

Анализ и исследование солнечного отопления при проектировании жилых зданий

Туголбай Матисаков
Кандидат технических наук, доцент
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0000-0001-8644-059X

Матисаков Жумакадыр
Кандидат технических наук, доцент
Ошский технологический университет
723503, ул. Н. Исанова, 81, г. Ош, Кыргызская Республика
https://orcid.org/0009-0000-7643-5324

Аннотация. Цель исследования заключалась в разработке математической модели для повышения энергоэффективности жилых зданий за счёт сезонного использования солнечного отопления. Предложено методику теоретического расчета к проектированию энергоактивных домов с учётом геометрических параметров окон и конструктивных особенностей оконной крыши. Выявлены условия оптимального размещения окон для эффективного улавливания солнечного излучения в течение отопительного сезона. Особое внимание в исследовании уделено климатическим особенностям Кыргызстана, таким как длительность отопительного сезона, уровень солнечной радиации и потенциал снижения потребления традиционных энергоносителей. Климатический анализ городов Ош и Бишкек показал, что даже в самые короткие зимние дни возможно получение значительного объёма солнечной энергии, достаточного для частичного или полного обогрева помещений. Разработана математическая модель теплопотерь, учитывающая разность температур между внутренней и наружной средой, а также коэффициент теплопередачи ограждающих конструкций. Это позволило оценить продолжительность эффективного использования солнечной энергии для отопления. Были формализованы ключевые параметры проектирования, включая угол падения солнечных лучей, длину свеса крыши, высоту окна и геометрию фасада. Также предложены формулы для расчёта длины и высоты козырька с учётом сезонных изменений положения солнца. В статье представлена схема размещения крыш и окон, обеспечивающая защиту от перегрева летом и максимальный приток солнечной энергии зимой. Исследование подтверждает, что грамотно спроектированные солнечные системы отопления могут значительно снизить нагрузку на центральное теплоснабжение и повысить эффективность использования возобновляемых источников энергии. Представленные методики применимы при проектировании современных энергоэффективных зданий, особенно в регионах с горным рельефом и длительным отопительным сезоном. Таким образом, данное исследование имеет большое значение для практической реализации систем солнечного отопления, способных обеспечить устойчивое и эффективное использование солнечной энергии с учётом местных климатических особенностей

Ключевые слова: возобновляемые источники энергии; тепловая эффективность; архитектурные решения; теплозащита зданий; климатические особенности; оптимизация строительных элементов; сезонная адаптация конструкции

VOL. 4, No. 1
Journal of Osh State University. Mathematics. Physics. Technical Sciences
2025
Journal of Osh State University
Journal homepage: https://mpht-oshsu.com/en
UDC 662.997.534
DOI: 10.52754/16948645_2025_4(1)_70
Article's History:
Received 29.01.2025 Revised 02.05.2025 Accepted 19.06.2025

Design of heat losses calculation method in elements of double-circuit solar water heating collectors

Sayit Sultanov*
PhD in Technical Sciences, Associate Professor
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0000-0003-4330-025X

Idirisbek Kenzhaev
Doctor of Technical Sciences, Professor
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
https://orcid.org/0000-0002-8094-2553

Zhanbolot Tursunbaev
PhD in Technical Sciences, Associate Professor
Osh Technological University
723503, 81 Isanov Str., Osh, Kyrgyz Republic
https://orcid.org/0009-0006-8465-7879

Ilyas Ryskulov
Senior Lecturer
Institute of Innovative Technologies and Energy
721405, 5 Studencheskaya Str., Tash-Kumyr, Kyrgyz Republic
https://orcid.org/0000-0002-5680-7554

Abstract. With the increasing interest in the use of renewable energy sources, in particular solar collectors, improving their energy efficiency has become particularly important. One of the key factors affecting the overall performance of such installations is heat losses through structural components. However, in practice, there is a lack of universal methods that allow for accurate assessment of these losses for collectors of different types. This determines the need to develop a flexible approach applicable to installations of different designs, which determines the relevance of this study. The aim of the study was to develop a method for calculating heat losses in elements of solar water heating collectors and to establish factors that directly affect its efficiency and performance. In these studies, computational and analytical research methods and thermodynamic analysis methods were used, and on their basis detailed information on heat losses in the collector elements was obtained. Based on the results of the conducted research, it was established that the main factors influencing the values of heat losses and the efficiency of double-circuit water heating collectors are the solar radiation density, the temperature of the environment and the working water. The obtained results make it possible to calculate the heat losses value through the construction elements of the collector. It was found that the greatest heat losses were observed from the collector's face covering. A heat balance equation was derived, and a thermal diagram of the solar water heating system was presented. Theoretically, changes in the heat transfer coefficient depending on ambient temperature and wind speed were investigated. The results obtained in the course of the study have scientific significance for further development and improvement of solar water-heating collector designs. In particular, the identified dependence of heat losses on the structural features of the collector's front side enables researchers and engineers to focus on its optimisation in order to reduce convective losses. This creates a foundation for the development of more effective engineering solutions in the design of such systems and can be used in the modelling, calculation, and testing of new solar collector designs

Keywords: solar radiation; heat transfer coefficient; heat exchange; heat balance; heat-absorbing surface; convection; radiation

Suggested Citation:
Sultanov S, Kenzhaev I, Tursunbaev Zh, Ryskulov I. Design of heat losses calculation method in elements of double-circuit solar water heating collectors. J Osh State Univ Math Phys Tech Sci. 2025;4(1):70–80. DOI: 10.52754/16948645_2025_4(1)_70

*Corresponding author

Introduction

The development and construction of power plants based on renewable energy sources is an important engineering task. Solar radiation is characterised by abundant resources, environmental friendliness, and availability, making it one of the most promising forms of renewable energy. Therefore, solar energy installations attract considerable interest. Considering this, the study of thermal characteristics of solar water heating collectors (SWC) and ways to improve their efficiency is significantly important.

The interest in solar water heating systems (SW) is growing due to their environmental friendliness and the possibility of saving energy costs. In a review by M.R. Al-Mamun *et al*. [1], modern SWC designs are analysed in detail, including flat and evacuated tube collectors, as well as the use of nanofluids to enhance heat transfer. The authors note the effectiveness of nanofluids (for example, MWCNT and Al_2O_3) in increasing efficiency by 28-35%, but point out the lack of models for double circuits, which emphasises the need to develop adaptive methods for calculating heat losses. In the study by A. Bouhdjar *et al*. [2], an improved method for calculating the total heat loss coefficient in flat-plate collectors is proposed, considering materials temperature dependence and thermal resistance. The authors presented an analytical model and experimentally confirmed its effectiveness for metal absorbers. However, the model is not adapted to complex geometries, such as tubular or double-loop structures, which requires further expansion.

The work of B.E. Khayriddinov *et al*. [3] was devoted to mathematical modelling of heat accumulation processes in double-loop flat solar water heating systems taking into account the stratification of the coolant. The authors developed an experimental module and compared the results of numerical modelling with practical observations, which made it possible to establish a relationship between the parameters of the heat exchanger (including the coil in the storage tank) and the efficiency of heat transfer. In the study particular attention is paid to the creation of a mathematical model of thermal accumulation using a system of equations implemented in a software environment. The data obtained demonstrated a high degree of agreement between theoretical and experimental results; however, the study is limited to stationary operating conditions and does not cover the influence of external climatic factors. R. Roy [4] focused on non-stationary thermal analysis of absorbers under conditions of changing solar radiation. The model developed in their study takes into account the thermal inertia of the material and allows predicting the point in time at which heat loss exceeds the useful heat flow. Despite its high theoretical accuracy, the work does not cover the influence of external factors such as wind and humidity, which limits its practical application in real installations. In the work of A.R. Kalair *et al.* [5], a numerical simulation of solar water heating systems was performed using various types of collectors, including flat, vacuum and parabolic concentrators. The authors showed that vacuum tube collectors have the highest seasonal efficiency, providing up to 50% coverage of hot water needs under favourable climatic conditions. The importance of selecting a collector design considers regional insolation and the thermal properties of the heat carrier is emphasised. N. Temirbaeva *et al*. [6] analysed the potential of solar energy in Kyrgyzstan and showed that the insolation level in the republic averages 6.4-6.7 kW h/m^2 per day, with more than 2,800 sunny hours per year. This makes solar energy particularly promising for autonomous and hybrid solutions. Despite the significant resource, the authors note the lack of effective engineering methods for calculating the performance of solar thermal systems taking into account local climatic conditions. J. Beringer's bachelor's thesis [7], completed at HAW Hamburg, contains an analysis of solar thermosyphon systems in rural areas of Kyrgyzstan. The author records high heat losses at night and insufficient insulation of pipelines, proposing to strengthen thermal protection and introduce bimetallic absorbers. The work is important from the point of view of regional applicability, but requires quantitative verification of the proposals.

In light of the above, the research and development of solar hot water supply systems using dual-circuit solar collectors represents a highly relevant and promising area of study. The aim of this work was to develop and validate a method for calculating the thermal parameters of a solar water heating system with a double-circuit collector, and to identify the factors that directly influence its efficiency and performance.

Materials and Methods

Heat exchange processes in solar water heating collectors (SWC) occur continuously, beginning from the moment when solar radiation reaches the surface of the collector, then converts into thermal energy and further heating of the coolant. These losses occur during the heat transfer process and directly affect the productivity and technical and economic indicators of the entire system.

The methodological basis of this study was the approaches developed in classical works by J.A. Duffy & W.A. Beckman [8], as well as R.R. Avezov [9], aimed at calculating heat losses in the SWC. In addition, the work relies on a series of earlier studies [10-11], in which the analysis of thermal characteristics was based on the assumption that incoming solar radiation is converted into useful heat spent on heating the intermediate coolant (water), while the remaining part is accumulated in the structural elements and subsequently lost into the environment.

A distinctive feature of the heat loss calculation method proposed in this paper is its adaptation to the custom design of a double-circuit solar water heating collector, developed by the authors and protected under patent KR No. 1706 [12]. The system differs from standard models in several structural and operational characteristics, which necessitated an individualised approach to modelling heat exchange processes.

To enable a more accurate assessment of heat losses, a thermal diagram was utilised, reflecting energy flows and the interaction of components within the system. The theoretical foundation for the analysis was a heat balance model, modified to account for the specific structural features of the studied installation. This approach enables more precise localisation of zones with the highest thermal losses and facilitates optimisation of the design for improved energy efficiency. The heat balance equation for solar water heating collectors during energy distribution is formulated as follows:

$$E = Q_p + Q_a + Q_{tp}, \tag{1}$$

where Q_p – useful energy expended to heat the consumed water in the solar collector; Q_a – energy accumulated in the main elements of solar collectors; Q_{tp} – heat losses from solar collector elements to the environment (through the top, sides and bottom). The efficiency of solar collectors is defined as:

$$\eta = \frac{Q_p}{E} = 1 - \frac{Q_a + Q_{tp}}{E}. \tag{2}$$

Consideration should be given to the heat accumulated in the components of solar collectors (Q_a) during the initial phase of system operation, or when significant fluctuations in solar radiation intensity occur throughout the day. However, when analysing the thermal performance of solar collectors at a specific moment in time, this component may be considered negligible. In such cases, the efficiency of the solar collectors can be expressed using the following equation:

$$\eta = \frac{Q_p}{E} = \frac{E - Q_{tp}}{E} = 1 - \frac{Q_{tp}}{E}. \tag{3}$$

The density of total solar radiation supplied to the surface of the water heating collector is determined as:

$$E = E_o FTA. \tag{4}$$

For double-glazing:

$$E = E_o FT^2 A, \tag{5}$$

$$Q_p = g(t_n - t_k), \tag{6}$$

$$Q_{tp} = Fk\Delta t, \tag{7}$$

where E_o – the density of total solar radiation falling on the surface of a solar collector; F – area of the solar collector's beam-receiving surface; T – light transmittance coefficient of glass coating; A – heat absorption coefficient of the heat-receiving surface; k – heat absorption coefficient of the heat-receiving surface; Δt – the difference between the indoor and outdoor air temperatures. The density of total solar radiation passing through single or double-layer upper glazing and reaching the heat-absorbing surface of a solar water heating collector, at:

$$E_0 = 0 \div 850\ \mathrm{W/m^2};$$
$$F_1 = 0.47\ \mathrm{m^2};\ F_2 = 0.4\ \mathrm{m^2};$$
$$T = 0.95;\ A = 0.95$$

is within $E = 0 \div 342\ \mathrm{W/m^2}$.

Heat losses from solar water heating collectors are generally defined as:

$$Q_{tp} = Q_a + Q_b + Q_c, \tag{8}$$

where Q_a – heat losses from the upper part of solar water heating collectors (from the glass cover side); Q_b – heat losses from the sides of solar water heating collectors; Q_c – heat losses from the bottom of solar water heating collectors.

To calculate heat losses in the structural elements of a solar water heating collector, heat balance equations were used, previously issued in the form of patents Patent No. 5284 [13], No. 5930 [14], Patent KR No. 1605 [15]. These equations are applicable to all types of installations created according to a similar scheme and examine the combined effects of heat transfer, energy storage and losses. Heat losses in the system were determined based on the following expression, reflecting the overall energy balance of the collector:

$$Q_{tp} = Q_a + Q_b + Q_c = \Delta t(k_a F_a + k_b F_b + k_c F_c), \tag{9}$$

where $k_a + k_b + k_c$ – heat transfer coefficients respectively through the top, sides and bottom of solar water heating collectors; $F_a + F_b + F_c$ – areas of the above-mentioned parts of solar water heating collectors.

To visualise heat exchange processes in the collector design, a thermal diagram developed by the authors was used, reflecting the interaction of the installation components, energy transfer and the direction of heat loss. Figure 1 shows the specified thermal diagram of a solar water heating collector with an intermediate water coolant. According to the given model, heat losses to the environment occur from the surface of all structural elements mainly due to convection and radiation.

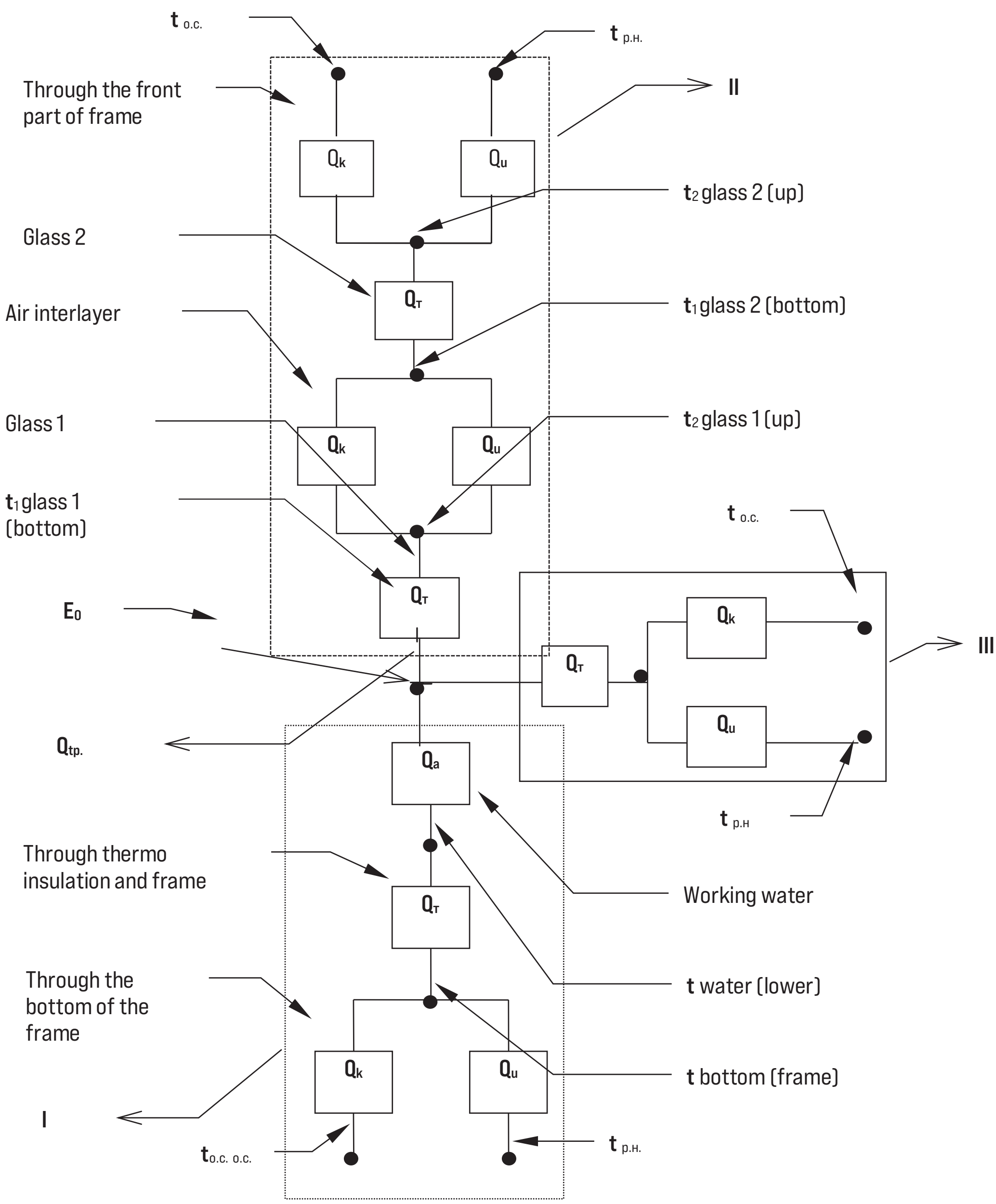

Figure 1. Thermal diagram of a solar water heating collector with an intermediate water coolant

Note: I, II, III – heat losses respectively through the bottom, upper part and side surfaces of the solar water heating collectors; Q_T, Q_K, Q_U – respectively heat flows transferred by thermal conductivity, convection and radiation; E_0 – density of incident solar radiation; Q_a – heat accumulated by the solar water heating collector; Q_{tp} – useful heat

Source: created by authors

The heat transfer coefficient through the upper part of solar water heating collectors is determined by the formula:

$$k_a = \left(\frac{1}{k_1+k_1^1} + \frac{1}{k_2+k_2^1}\right)^{-1}, \tag{10}$$

where k_1 – coefficient of convective heat transfer between glass coatings in a solar water heating collector; k_1^1 – coefficient of radiant heat transfer between glass coatings; k_2 – coefficient of convective heat exchange between the top glass covering and the environment; k_2^1 – coefficient of heat transfer by radiation from a glass covering to the sky. In turn k_1 is determined by formula:

$$k_1 = \frac{k_{T_g}}{k_{10}} = 1 - 0.0018(T_{av} - 10), \tag{11}$$

where $k_{10} = 1.14\frac{\Delta T^{0.31}}{l^{0.07}}$ – empirical formula for determining the coefficient of convective heat exchange between the glass coatings of a solar water heating collector, in the case where the temperature difference between the outer and inner surfaces of the solar water heating collector is ~ 10 °C, which is an acceptable value. Then:

$$k_1 = \left[1 - 0.0018(T_{av} - 10)\right] \times 1.14 \frac{\Delta T^{0.31}}{l^{0.07}}, \quad (12)$$

where l – distance between glass coverings; T_{av} – average temperature of the outer part of the heat-absorbing surface and the average inner surface of the outer glass covering; ΔT – temperature difference between the average outer part of the heat-absorbing surface and the inner surface of the outer glass covering.

In this case, the inner glass coating is in direct contact with the polyethylene film of the container, and it can be considered that both of them simultaneously perform the function of the heat-absorbing surface. The coefficient of heat transfer by radiation from the heat-absorbing surface to the outer glass coating, located at a distance of 30 mm, is determined by the expression:

$$k_1^1 = \frac{\sigma(T_p + T_g)(T_p^2 + T_g^2)}{\frac{1}{\varepsilon n} + \frac{1}{\varepsilon g} - 1}, \quad (13)$$

where σ – is the Stefan-Boltzmann constant; T_p – temperature of the heat-absorbing surface of the solar water-heating collector; T_g – temperature of the inner surface of the outer glass covering; is the emissivity of the heat-absorbing polyethylene film; ε_n – emissivity of the glass covering of the solar water-heating collector.

The convective heat transfer coefficient from the surface of the solar water heating collector to the surrounding environment depends on the air velocity (V_a) and is determined by the following equation:

$$k_2 = 5.7 + 3.8\,V_a. \quad (14)$$

The coefficient of heat transfer by radiation from the outer glass surface of a solar water heating collector to the sky is determined by the expression:

$$k_2^1 = \varepsilon_g \sigma (T_o + T_n)(T_o^2 + T_n^2), \quad (15)$$

where T_0 – ambient temperature; $T_n = T_0 - 6$ – radiation temperature of the sky.

In this case, all parameter values are known except for the temperature of the outer surface of the inner glass coating (T_g), for which the iteration method was used. First, the temperature value was accepted and the heat transfer coefficient value was determined at the top of the solar water heating collector. To determine the temperature value of the inner surface of the outer glass coating, the following formula was used:

$$T_g = T_p - \frac{k_a(T_p - T_o)}{k_1 + k_1^1}. \quad (16)$$

In this case, the T_g temperature values are determined for the following values of the solar water heating collector parameters: $l = 0.03$m; $T_{av} = 58\,^\circ$C; $\Delta T = 44\,^\circ$C; $T_p = 80\,^\circ$C; $\varepsilon_n = 0.95$; $\varepsilon_g = 0.88$; $T_o = 35\,^\circ$C; $T_n = 29\,^\circ$C.

Results and Discussion

Using formulas (12)-(15), the numerical values of the coefficients k_1, k_1^1, k_2, k_2 were determined. Substituting the values of the named coefficients into formula (10) and carrying out the corresponding calculations, it was established that the values of the heat transfer coefficient from the bottom, side and top parts of the solar water heating collector at wind speed $V_a = 0 \div 6$ m/sec are $k_a = 1.286 \div 2.865$ W/(m^{2}deg).

For the side and bottom parts of the solar water heating collector, the heat transfer coefficient values are the same and are determined by the expression:

$$k_b = k_c = \frac{1}{\frac{\delta_p}{\lambda_p} + \frac{\delta_i}{\lambda_i} + \frac{\delta_c}{\lambda_c}} + \frac{1}{k_2 + k_2^1}, \quad (17)$$

where δ_p and λ_p – are the thickness and thermal conductivity coefficient of the polyethylene film, respectively; δ_i and λ_i – are the same for thermal insulation; δ_c and λ_c – are the same for the cinder concrete wall of the solar water heating collector housing $p = 1{,}500$ kg/m^3.

By substituting the numerical values of the parameters included in equations (17) and the calculations performed, the numerical values of the heat transfer coefficient k_b through the sides and bottom of the solar water heating collector housing were determined, which amounted to $k_b = k_c = 0.82 \div 2.14$ W/(m^{2}deg).

The results of the calculations performed using formulas (10) and (17) are presented in Table 1. The calculations were performed for different temperatures of the heated water, taking into account the change in wind speed. The numerical values of the heat transfer coefficients from the bottom, side and upper parts of the solar water heating collector were determined.

Table 1. Heat transfer coefficients depending on wind speed and working water temperature k_a, k_b, k_c W/(m^{2}deg)

No.	Temperature of working water in SWH, °C	Wind speed, m/sec						
		0	1	2	3	4	5	6
1.For the top of the solar water heating collectors	40	1.286	1.413	1.476	1.513	1.538	1.555	1.569
	50	1.647	1.860	1.970	2.040	2.084	2.116	2.141
	60	1.830	2.100	2.242	2.328	2.387	2.430	2.463
	70	1.950	2.260	2.426	2.527	2.600	2.650	2.687
	80	2.045	2.385	2.570	2.685	2.764	2.820	2.865

Table 1. Continued

No.	Temperature of working water in SWH, ˚C	Wind speed, m/sec						
		0	1	2	3	4	5	6
2. For bottom and sides solar water heating collectors	40	0.82	0.88	0.98	1.19	1.31	1.43	1.58
	50	0.96	1.05	1.15	1.26	1.38	1.50	1.63
	60	1.10	1.19	1.31	1.43	1.58	1.75	1.94
	70	1.22	1.31	1.45	1.62	1.76	1.90	2.06
	80	1.31	1.44	1.58	1.74	1.80	1.97	2.14

Source: created by authors

Analysis of the heat transfer coefficient values at the bottom, sides, and top of the solar water heating collector, as presented in Table 1, indicates that an increase in ambient wind speed leads to a corresponding rise in these coefficients. It is also noteworthy that higher temperatures of the working water result in a more pronounced increase in heat transfer coefficient values.

By substituting the obtained coefficients corresponding to different water temperatures into equation (9), the heat losses through the structural components of the solar water heating collectors were calculated. Table 2 presents the results of these calculations, showing total heat losses as a function of air velocity and working water temperature.

It was found that for a solar water heating collector with an area of $F_a = 0.47$ m^2, $F_b + F_c = 0.531$ m^2 total heat losses according to the calculations were within the range: $Q_{tp} = 9.4 \div 189.5$ W [12].

Table 2. Heat losses from the bottom, sides and top of solar water heating collectors depending on wind speed and working water temperature Q_a, Q_b, Q_c, W

Main elements of the collector	Temperature of working water in SWH, ˚C	Wind speed, m/sec						
		0	1	2	3	4	5	6
1. For the top of solar water heating collectors, Q_a	40	3.4	3.6	3.7	3.8	3.8	3.8	3.8
	50	12.3	13.3	13.9	14.4	14.8	15.0	15.2
	60	24.1	26.4	27.8	28.6	29.2	30.3	30.6
	70	32.6	35.8	37.6	38.4	39.2	40.0	41.0
	80	40.8	44.6	49.4	52.1	53.6	55.1	56.0
2. For side parts of solar water heating collectors, Q_b	40	0.6	0.7	0.7	0.8	0.8	0.8	0.8
	50	1.8	1.9	2.0	2.1	2.2	2.3	2.4
	60	3.2	3.4	3.5	3.6	3.7	3.8	3.8
	70	5.4	6.8	7.6	8.2	8.5	8.7	8.9
	80	6.8	8.4	10.2	11.8	12.4	13.2	14.0
3. For the bottom of solar water heating collector housings, Q_c	40	5.4	6.1	6.4	6.5	6.7	6.8	6.9
	50	22.2	25.8	28.4	28.9	29.8	30.1	32.2
	60	39.7	47.3	51.1	53.3	54.9	55.2	57.4
	70	62.0	74.6	80.6	83.4	86.2	88.6	89.8
	80	87.8	104.7	110.8	114.4	116.8	118.2	119.5

Source: created by authors

Based on the results presented in Tables 1 and 2, it can be concluded that the heat transfer coefficient at the bottom, sides, and top of the solar water heating collector depends on both the working water temperature and wind speed. Specifically, as these parameters increase, the heat transfer coefficients of the collector's protective components also rise. These findings confirm that the heat exchange between the collector and the surrounding environment becomes more pronounced with higher water temperatures and varying external conditions.

The maximum heat transfer coefficient for the upper part of the collector using an intermediate water coolant was calculated as 2.865 W/(m^{2}deg), while the corresponding values for the bottom and side sections were 2.14 W/(m^{2}deg). The resulting maximum heat losses were as follows: 119.5 W from the upper part with a surface area of 0.47 m^2; 14 W from the side walls with an area of 0.101 m^2; and 56 W from the bottom with an area of 0.43 m^2.

For the given collector geometry and operating conditions – maximum solar radiation intensity, working water temperature in the polyethylene container, and wind speed – the total heat loss was estimated at 189.5 W. The results also showed that as the intensity of total solar radiation increased, the temperature of the working fluid rose accordingly. This, in turn, led to a greater temperature difference between the fluid and ambient air, thereby increasing heat losses from the collector's structural elements.

Given that heat losses depend on both the working water temperature and the solar radiation intensity, the efficiency of the solar water heating collector was calculated using equations (3) and (4). During the experiments, the maximum solar radiation intensity measured on the collector surface reached 860 W/m^2. Based on the obtained efficiency values, graphical dependencies were constructed to illustrate efficiency as a function of solar radiation intensity and ambient temperature at an air speed of 3 m/s (Fig. 2).

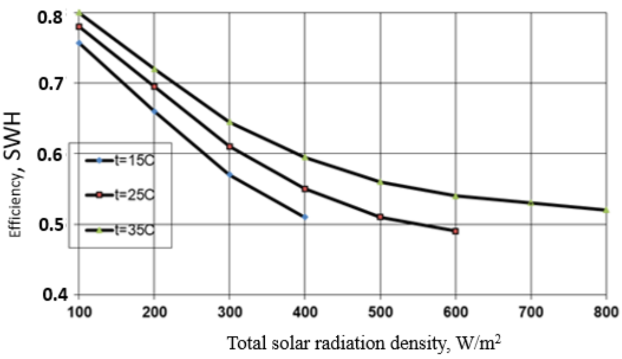

Figure 2. Dependence of the efficiency value of a solar water heating collector with an intermediate water coolant on the density of the total solar radiation and the ambient temperature
Source: compiled by the authors

It is evident from Figure 2 that as the density of total solar radiation increases, the efficiency decreases depending on the ambient temperature. This suggests that with an increase in the temperature of the working fluid in the solar water heating collector, heat losses increase correspondingly, leading to a decrease in overall efficiency.

The data obtained from the calculations enable a more detailed assessment of heat transfer and heat loss characteristics in the design of double-circuit solar water heating collectors (SWC) with an intermediate water coolant. Analysis of the results revealed that increases in working fluid temperature and wind speed lead to higher heat transfer coefficients, thereby resulting in greater heat losses through the collector body. The upper and lower sections of the collector were most sensitive to external factors, whereas the side walls demonstrated greater stability. This highlights the necessity for improved thermal insulation in specific zones of the SWC during the design stage. Furthermore, calculations of the efficiency coefficient (EC) showed that an increase in solar radiation intensity is accompanied by a decrease in collector efficiency, due to increased heat losses at higher fluid temperatures.

Comparison with other studies supports the reliability of these findings. For example, in the work by Sh.I. Klychev *et al*. [16], the heat losses of a three-layer underground cylindrical heat accumulator used in solar installations were examined. The authors demonstrated that the efficiency of heat storage is significantly influenced by the thermal properties of each layer, the foundation depth, and system operating time. The greatest losses occurred through the lower layer in contact with the ground, while the internal layer ensured more stable heat retention. The developed model enabled the calculation of temperature distribution over time and space, which is vital for optimising long-term solar heating systems.

The data obtained are particularly relevant for regions with high solar radiation, such as Kyrgyzstan. According to E. Dyikanov [17], the average annual sunshine duration in the country is 2,800-3,000 hours, and the solar radiation level reaches 6.5 kWh/m^2 per day, making the application of solar water heating systems highly promising. Given that hot water supply in the public sector accounts for up to 20% of total energy consumption, enhancing the efficiency of SWCs – by reducing heat losses and optimising design – can significantly contribute to energy conservation and lower operational costs.

The calculation results were compared with data from several previously published studies. In the work by W. Beckman *et al*. [18], a classical methodology for estimating heat losses in solar collectors was proposed, based on heat balance equations with differential analysis of heat transfer through transparent covers, side walls, and the collector base. According to their findings, at an average water temperature of 60°C, heat losses through glazing reached approximately 30-33 W/m^2, and total losses under intense solar radiation amounted to 90-100 W. In the present study, a similar method was adapted for a dual-circuit system with an intermediate coolant, allowing for additional losses associated with material thermal inertia and temperature stratification to be considered. Under these conditions, maximum losses from the upper part of the collector reached 119.5 W, exceeding the values in W. Beckman *et al.* model, which can be attributed to the complexity of the design and the presence of an additional heat exchange circuit.

In the study by D.M. Rakhimov [19], optimisation conditions for a single-circuit solar installation with intensive water heating were analysed. It was shown that as the coolant temperature exceeded 70°C, heat losses rose sharply, reaching 140-150 W in cases of insufficient insulation. The observed patterns align well with those of the present study, which similarly established the influence of working fluid temperature and wind speed on heat transfer coefficient growth and subsequent heat loss. The key difference lies in the system configuration: Rakhimov examined a direct heating setup, whereas this study focuses on a two-circuit scheme, which ensures more stable temperatures due to the intermediate fluid.

In the work of T.T. Omorov & D.M. Rakhimov [20], a low-inertia solar installation was developed that could reach steady-state conditions within 10-12 minutes at a solar radiation intensity of 850 W/m^2. Their calculations indicated a system efficiency of 62%, with minimal heat losses due to the absence of thermal storage components. In contrast, the present study employed a solar water heating collector with an intermediate coolant, which required more than 20 minutes to reach thermal equilibrium. This is attributed to the need to heat both the intermediate fluid and the massive structural elements, resulting in increased thermal inertia and slower system response to external fluctuations.

The mathematical model proposed by J.J. Tursunbaev *et al*. [21] described a solar installation operating on natural circulation via the siphon effect. The model incorporated parameters such as gravitational pressure and hydrodynamic resistance, achieving high energy efficiency with minimal power consumption. Maximum losses from the upper section of the unit were approximately 95 W at a water temperature of 75°C. Unlike the present study, that model did not involve forced circulation or account for stratification and thermal storage, both of which are key components of the current system. Consequently, the energy stability and application range of Tursunbaev's model are limited to specific non-pumped conditions, whereas the collector examined here demonstrated consistent performance across a broader range of loads and environmental conditions.

In a review by F. Eze *et al*. [22], modern solar water heating technologies were evaluated with a focus on design parameters influencing heat loss and cost efficiency. Key factors included the type of glazing, insulation thickness and properties, and overall system geometry. The authors concluded that employing double glazing and polyurethane insulation could reduce heat loss to 18-20% of total heat flux. These conclusions are corroborated by the present study, where the highest heat losses were recorded at the upper (119.5 W) and lower (56 W) parts of the installation, which utilised multilayer materials with varying thermal conductivities. The alignment of results underscores the significance of structural insulation in enhancing solar collector efficiency.

Conclusions

In this study, a method was developed to determine heat losses through the structural components of double-circuit solar water heating collectors, taking into account their specific design features. The authors proposed a thermal model that enables a detailed analysis of heat exchange processes and the direction of heat losses within the system, significantly simplifying the modelling of heat flows and the assessment of collector efficiency. Based on this model, heat transfer coefficients were calculated for the upper, side, and bottom parts of the structure at various temperatures of the working fluid and wind speeds.

The conducted investigations revealed that in double-circuit installations, the intermediate water coolant is heated first and subsequently transfers thermal energy to the consumed water, thus providing greater thermal inertia compared to conventional single-circuit collectors. Convective heat exchange between the coolant and the heat exchanger determines the rate of heat transfer and is a key factor in the overall efficiency of the system.

The calculations established that the greatest heat losses occurred through the collector's front cover, with values 2 to 2.5 times higher than those from the bottom and 8 to 9 times higher than those from the side walls. The maximum heat transfer coefficients were found to be 2.865 W/(m^2deg) for the upper part and 2.14 W/(m^2deg) for the side and bottom parts. Analysis of the dependence of these coefficients on fluid temperature and wind speed demonstrated a natural increase in losses as these parameters rise.

Thus, the proposed methodology and thermal model enable a comprehensive evaluation of the thermal performance of double-circuit solar water heaters and identify key strategies for minimising heat loss. Future research should focus on experimental validation of the methodology, the development of advanced thermal insulation materials, and the adaptation of the model to varying climatic conditions to enhance system energy efficiency.

Acknowledgements

This research article was prepared as part of the ERASMUS+ SILKWAY Project (No. 101179683), which aims to generate electricity and thermal energy from renewable energy sources.

Funding

None.

Conflict of Interest

None.

References

[1] Al-Mamun MR, Roy H, Islam MS, Ali MR, Hossain MI, Aly MAS, et al. State-of-the-art in solar water heating (SWH) systems for sustainable solar energy utilization: A comprehensive review. Sol Energy. 2023;264:111998. DOI: 10.1016/j.solener.2023.111998

[2] Bouhdjar A, Semai H, Amari A. New technique to evaluate the overall heat loss coefficient for a flat plate solar collector. J Energy Technol. 2023;14(1):11–25. DOI: 10.18690/jet.14.1.11-25.2021

[3] Khayriddinov BE, Ergashev SH, Ganiyev SYu. Mathematical modeling of the heat accumulation system in a hot water collector using solar energy. Int J Adv Res Sci Eng Technol. 2023;10(8):20918–23.

[4] Roy R. Appropriate transient thermal analysis of an absorber plate in flat-plate solar collectors from beginning to end operational conditions. J Mod Thermodyn Mech Syst. 2022;4(1):12–37.

[5] Kalair AR, Seyedmahmoudian M, Saleem MS, Abas N, Rauf S, Stojcevski A. A comparative thermal performance assessment of various solar collectors for domestic water heating. Int J Photoenergy. 2022;(1):9536772. DOI: 10.1155/2022/9536772
[6] Temirbaeva N, Sadykov M, Osmonov Zh, Osmonov Y, Karasartov, U. Renewable energy sources in Kyrgyzstan and energy supply to rural consumers. Mach Energy. 2024;15(3):22–32. DOI: 10.31548/machinery/3.2024.22
[7] Beringer J. Solar thermal systems for space heating in cold climates: A validated case study from Kyrgyzstan [Bachelor thesis]. Ingolstadt: Technische Hochschule Ingolstadt; 2022.
[8] Duffy JA, Beckman WA. Thermal processes using solar energy. Moscow: Mir; 1997. 420 P.
[9] Avezov RR. Solar heating and hot water supply systems. Tashkent: Fan; 2008. 288 P.
[10] Ismanzhanov AI, Sultanov SK. Study of the technical and economic characteristics of solar water-heating installations made from alternative materials. Appl Sol Energy. 1999.
[11] Ismanzhanov AI, Sultanov SK. Comparative operating characteristics of solar water-heating collectors. Appl Sol Energy. 2001.
[12] Patent KR No. 1706. Solar water heating installation ISR-1. Bishkek, Kyrgyzstan; 2014. 6 P.
[13] Patent No. 5284. Solar water heater. Republic of Uzbekistan; 1998. 4 P.
[14] Patent No. 5930. Solar water-heating collector. Republic of Uzbekistan; 1999. 4 P.
[15] Patent KR No. 1605. Solar water heating installation. Bishkek, Kyrgyzstan; 2012. 4 P.
[16] Klychev ShI, Kenzhaev IG, Bagyshev AS, Bakhramov SA, Kadyrgulov DE. Thermal losses of a three-layer underground cylindrical heat accumulator of solar installations. Appl Sol Energy. 2020;57(6):523–7. DOI: 10.3103/S0003701X21060116
[17] Dyikanov E. Renewable energy and energy efficiency in the Kyrgyz Republic [Internet]. 2013 Oct 7 [cited 2025 March 7]. Available from: https://www.energycharter.org/fileadmin/DocumentsMedia/Events/20131007-9RECA_S2_EDyikanov_ru.pdf
[18] Beckman WS, Klein J, Duffy J. Calculation of solar heating systems. Moscow: Energoizdat; 1982. 280 P.
[19] Rakhimov DM. (2024). Optimisation of parameters of solar water heating installation with intensive water heating. Res Focus Int Sci J. 2023;2(12):22–6.
[20] Omorov TT, Rakhimov DM. Development and research of low-inertia solar water heating plant. Int Sci J. 2023;2(4):58–63.
[21] Tursunbaev JJ, Matisakov TK, Ergashov MO. Mathematical model for solar heat supply system with siphon effect. Proc Univ Kyrgyzstan. 2023;(5).
[22] Eze F, Egbo M, Anuta UJ, Ntiriwaa OB, Ogola J, Mwabora J. A review on solar water heating technology: Impacts of parameters and techno-economic studies. Bull Natl Res Cent. 2024;48:29. DOI: 10.1186/s42269-024-01187-1

Эки контурлуу күн суу жылытуу коллекторлорунун элементтериндеги жылуулук жоготууларын эсептөө методологиясын иштеп чыгуу

Сайит Султанов
Техника илимдеринин кандидаты, доцент
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0000-0003-4330-025X

Идирисбек Кенжаев
Техникалык илимдердин доктору, профессор
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
https://orcid.org/0000-0002-8094-2553

Жанболот Турсунбаев
Техника илимдеринин кандидаты, доцент
Ош технологиялык университети
723503, Н. Исанов көч., 81, Ош ш., Кыргыз Республикасы
https://orcid.org/0009-0006-8465-7879

Ильяс Рыскулов
Улук окутуучу
Инновациялык технологиялар жана энергетика институту
721405, Студент көч., 5, Таш-Көмүр ш., Кыргыз Республикасы
https://orcid.org/0000-0002-5680-7554

Аннотация. Энергиянын кайра жаралуучу булактарын, атап айтканда, күн коллекторлорун пайдаланууга кызыгуунун жогорулашынын шартында алардын энергетикалык натыйжалуулугун жогорулатуу милдети өзгөчө мааниге ээ болууда. Мындай орнотмолордун жалпы иштешине таасир этүүчү негизги факторлордун бири конструкциялык элементтер аркылуу жылуулукту жоготуу болуп саналат. Бирок иш жүзүндө ар кандай типтеги коллекторлор үчүн бул жоготууларга так баа берүүгө мүмкүндүк берген универсалдуу методдор жетишсиз. Бул ар кандай конструкциядагы орнотууларга колдонулуучу ийкемдүү ыкманы иштеп чыгууну талап кылат, бул изилдөөнүн актуалдуулугун аныктайт. Изилдөөнүн максаты күн суу жылытуу коллекторлорунун элементтериндеги жылуулук жоготууларын эсептөө методологиясын иштеп чыгуу жана анын эффективдүүлүгүнө жана натыйжалуулугуна түздөн-түз таасир этүүчү факторлорду аныктоо болгон. изилдөөлөр эсептөө жана аналитикалык изилдөө ыкмаларын жана термодинамикалык талдоо ыкмаларын колдонгон; алардын негизинде коллектордук элементтердеги жылуулук жоготуулары боюнча толук маалымат алынган. Изилдөөлөрдүн натыйжалары боюнча, жылуулук жоготууларынын маанилерине жана кош контурлуу суу жылытуу коллекторлорунун эффективдүүлүгүнө таасир этүүчү негизги факторлор күн радиациясынын тыгыздыгы, айлана-чөйрөнүн температурасы жана жумушчу суунун экендиги аныкталган. Алынган натыйжалар коллектордун структуралык элементтери аркылуу жылуулук жоготууларынын маанилерин аныктоого мүмкүндүк берет. Жылуулуктун эң чоң жоготуулары коллектордун бет капталынан байкалаары аныкталган. Жылуулук балансынын теңдемеси түзүлүп, сууну жылытуу үчүн күн коллекторлорунун жылуулук диаграммасы келтирилген. Айлана-чөйрөнүн температурасына жана шамалдын ылдамдыгына жараша жылуулук берүү коэффициентинин өзгөрүүсү теориялык жактан изилденген. Изилдеенун журушунде алынган натыйжалар сууну жылытуучу кун коллекторлорунун конструкцияларын мындан ары иштеп чыгуу жана еркундетуу учун илимий мааниге ээ. Атап айтканда, жылуулук жоготууларынын коллектордун бетинин структуралык өзгөчөлүктөрүнөн көз карандылыгы конвективдик жоготууларды азайтуу үчүн аны оптималдаштырууга көңүл бурууга мүмкүндүк берет. Бул мындай установкаларды долбоорлоодо кыйла натыйжалуу инженердик чечимдерди иштеп чыгуу үчүн негиз түзөт жана күн коллекторлорунун жаңы конструкцияларын моделдөө, эсептөөлөр жана сыноолордо колдонулушу мүмкүн

Негизги сөздөр: күн радиациясы; жылуулук берүү коэффициенти; жылуулук алмашуу; жылуулук балансы; жылуулук кабыл алуучу бети; конвекция; радиация

Разработка методики расчета тепловых потерь в элементах двухконтурных солнечных водонагревательных коллекторов

Сайит Султанов
Кандидат технических наук, доцент
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0000-0003-4330-025X

Идирисбек Кенжаев
Доктор технических наук, профессор
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
https://orcid.org/0000-0002-8094-2553

Жанболот Турсунбаев
Кандидат технических наук, доцент
Ошский технологический университет
723503, ул. Н. Исанова, 81, г. Ош, Кыргызская Республика
https://orcid.org/0009-0006-8465-7879

Ильяс Рыскулов
Старший преподаватель
Институт инновационных технологий и энергетики
721405, ул. Студенческая, 5, г. Таш-кумыр, Кыргызская Республика
https://orcid.org/0000-0002-5680-7554

Аннотация. В условиях увеличения интереса к использованию возобновляемых источников энергии, в частности солнечных коллекторов, особенно важной становится задача повышения их энергетической эффективности. Одним из ключевых факторов, влияющих на общую производительность таких установок, являются тепловые потери через конструктивные элементы. Однако на практике существует недостаток универсальных методик, позволяющих точно оценивать эти потери для коллекторов различных типов. Это обуславливает необходимость разработки гибкого подхода, применимого к установкам разной конструкции, что и определяет актуальность настоящего исследования. Целью проведенного исследования являлась разработка методики расчета тепловых потерь в элементах солнечных водонагревательных коллекторов и установление факторов, непосредственно влияющих на его эффективность и производительность работы. В исследованиях использовались расчетно-аналитические методы исследования и методы термодинамического анализа на их основе получена подробная информация о тепловых потерях в элементах коллектора. В результате проведённых исследований установлено, что основными факторами, влияющими на значения тепловых потерь и на коэффициент полезного действия двухконтурных водонагревательных коллекторов являются плотность солнечного излучения, температуры окружающей среды и рабочей воды. Получены результаты, позволяющие определять значения тепловых потерь через конструктивные элементы коллектора. При этом установлено, что наибольшие тепловые потери наблюдались со стороны лицевого покрытия коллектора. Составлено уравнение теплового баланса и приведена тепловая схема солнечных коллекторов для нагрева воды. Теоретически исследованы изменения коэффициента теплопередачи в зависимости от температуры окружающей среды и скорости ветра. Полученные в ходе исследования результаты представляют научную значимость для дальнейших разработок и совершенствования конструкций солнечных водонагревательных коллекторов. В частности, выявленная зависимость тепловых потерь от конструктивных особенностей лицевой стороны коллектора позволяет сосредоточить внимание на её оптимизации с целью снижения конвективных потерь. Это создаёт основу для разработки более эффективных инженерных решений при проектировании подобных установок и может быть использовано при моделировании, расчётах и испытаниях новых конструкций солнечных коллекторов

Ключевые слова: солнечная радиация; коэффициент теплопередачи; теплообмен; тепловой баланс; тепловоспринимающая поверхность; конвекция; излучения

Journal of Osh State University. Mathematics. Physics. Technical Sciences

Scientific Journal

Vol. 4, No. 1, 2025

Publishing Address:
Osh State University
723500, 331 Lenin Str., Osh, Kyrgyz Republic
E-mail: info@mpht-oshsu.com
https://mpht-oshsu.com/en

Ош мамлекеттик университетинин Жарчысы. Математика. Физика. Техника

Илимий журнал

Том 4 №1 2025

Башкы редактор:
Адахимжан Сопуев

Редакциянын дареги:
Ош мамлекеттик университети
723500, Ленин көч., 331, Ош ш., Кыргыз Республикасы
E-mail: info@mpht-oshsu.com
https://mpht-oshsu.com/ky

Вестник Ошского государственного университета. Математика. Физика. Техника

Научный журнал

Том 4 №1 2025

Адрес редакции:
Ошский государственный университет
723500, ул. Ленина, 331, г. Ош, Кыргызская Республика
E-mail: info@mpht-oshsu.com
https://mpht-oshsu.com/ru